Higher Finiteness Properties of Reductive Arithmetic Groups in Positive Characteristic: the Rank Theorem

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Abstract

We show that the finiteness length of an S-arithmetic subgroup Γ in a non-commutative isotropic absolutely almost simple group $\mathcal{G}$ over a global function field is one less than the sum of the local ranks of $\mathcal{G}$ taken over the places in S. This determines the finiteness properties for arithmetic subgroups in isotropic reductive groups, confirming the conjectured finiteness properties for this class of groups.

Our main tool is Behr-Harder reduction theory which we recast in terms of the metric structure of euclidean buildings.

Let K be a global function field and let $\mathcal{G}$ be a linear algebraic group defined over K. We fix a finite set S of places over K and let $\mathcal{O}_S$ denote the subring of S-integers in K. We want to talk about the group $\Gamma := \mathcal{G}(\mathcal{O}_S)$, but as an algebraic variety, the R-points of $\mathcal{G}$ are only well-defined for K-algebras R. However, we regard $\mathcal{G}$ as a concrete matrix group defined by polynomial equations in the matrix coefficients. That is, we choose a particular realization of the variety $\mathcal{G}$ as an algebraic set in some affine space. Given this realization, we define Γ as its set of $\mathcal{O}_S$-points. The subgroup Γ obtained this way is called an <u>S-arithmetic subgroup</u> of $\mathcal{G}$. Of course, the arithmetic group Γ depends on the chosen realization of $\mathcal{G}$, but any two choices lead to S-arithmetic subgroups of $\mathcal{G}$ that share a common subgroup of finite index in both; see, e.g., [Serr79, §1]. Hence, the commensurability class of Γ depends only on the group scheme $\mathcal{G}$ and the S-arithmetic ring $\mathcal{O}_S$.

We are interested in finiteness properties of the group Γ. Recall that a group G is of <u>type F_m</u> if it admits a classifying space with finite m-skeleton. The <u>finiteness length $\phi(G)$ of G</u> is the largest m such that G is of type F_m. We say that $\phi(G) = \infty$ if G is of type F_m for all m. The finiteness length is a commensurability invariant: in fact, it is invariant under quasi-isometries [Alon94]. In particular, the finiteness length $\phi(\Gamma)$ of the S-arithmetic group Γ depends only on $\mathcal{G}$ and $\mathcal{O}_S$ but not on the particular chosen realization of $\mathcal{G}$ as a matrix group.

Let K_p be the completion of the field K at the place p. The <u>local rank</u> of $\mathcal{G}$ at the place p is the rank of $\mathcal{G}$ over the field K_p. If $\mathcal{G}$ is isotropic and absolutely

almost simple, the group $\mathcal{G}(K_p)$ acts on its associated Bruhat-Tits building X_p and the dimension $\dim(X_p)$ is the local rank of $\mathcal{G}$ at the place p. We prove the following theorem, which answers [AbBr08, Question 13.20].

Rank Theorem. *Let $\mathcal{G}$ be a connected non-commutative absolutely almost simple K-isotropic K-group. Then the finiteness length $\phi(\Gamma)$ of the S-arithmetic group $\Gamma = \mathcal{G}(\mathcal{O}_S)$ is $d-1$ where $d := \sum_{p\in S}\dim(X_p)$ is the sum of the local ranks of $\mathcal{G}$.*

It was shown in [BW07] that $\phi(\Gamma) \leq d-1$. Hence, only the positive statement is new: Γ is of type F_{d-1}.

If $\mathcal{G}$ is reductive and anisotropic, J.-P. Serre showed that S-arithmetic subgroups are of type F_∞. More precisely, $\mathcal{G}(\mathcal{O}_S)$ has a torsion free subgroup of finite index that admits a finite Eilenberg-Mac Lane complex [Serr71, Cas (b), p. 126–127].

The Rank Theorem contrasts with the number field case where S-arithmetic subgroups of reductive groups are of type F_∞ [BoSe76, § 11]. We refer to the introduction of [BW07] for some conjectures about a more quantitative account which should reveal indeed deep similarities in the geometric underpinnings of both cases.

Interest in finiteness properties of Γ started in 1959 when H. Nagao [Naga59] showed that $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not finitely generated. In this case, there is a single place and the corresponding euclidean building is a tree, thus $d = 1$.

In 1969, H. Behr [Behr69] proved that Γ is finitely generated if and only if $d > 1$. He had to exclude a few cases. However, as he pointed out, those restrictions can be removed by appealing to Harder's version [Hard69] of reduction theory. Using Harder's reduction theory again, Behr [Behr98] showed in 1998 that Γ is finitely presented if and only if $d > 2$.

Concerning higher finiteness properties (i.e., beyond finite presentability), U. Stuhler [Stuh80] showed that $\mathrm{SL}_2(\mathcal{O}_S)$ has finiteness length $|S|-1 = d-1$. H. Abels [Abel91] and P. Abramenko [Abra87] showed that $\mathrm{SL}_n(\mathbb{F}_q[t])$ have finiteness length $n-2 = d-1$ provided that q is large enough depending on n. Abramenko [Abra96] extended this result to classical groups by recasting it in the context of groups acting on twin buildings. The need to exclude small q arises from the method of proof: it involves the analysis of certain subcomplexes in spherical buildings which only have sufficient topological connectivity if the underlying buildings are sufficiently thick. The articles [DGM09] and [GrWi09] also suffer from this shortcoming as they use a filtration modeled upon the filtration by combinatorial codistance introduced by Abels and Abramenko. Hence, similar relative links arise.

In 2005, Bernd Schulz analyzed in his PhD thesis [Schu10] a class of subcomplexes of spherical buildings which have the right topological connectivity without restrictions. Immediately following his discovery, several results were obtained in the positive direction (i.e., establishing that Γ is of type F_{d-1}). In [BW08], K. Wortman and the first author have proved the Rank Theorem for K-groups of K-rank one using reduction theory as a source for a filtration that leads to relative links analyzed by Schulz. In a previous version of this paper [BGW09], the authors eliminated the restriction on q in Abramenko's results. The third author [Witz10] extended the analysis to $\mathcal{G}(\mathbb{F}_q[t,t^{-1}])$. The basic idea was to replace combinatorial codistance by a metric codistance that leads to better behaved relative links. This paper extends

the reduction theory approach from [BW08] and removes the restriction on the global rank.

The Rank Theorem allows one to deduce finiteness properties of arbitrary reductive groups as described in [Behr98, 2.6(c), page 91]: First pass to the connected component of the identity. For the arithmetic subgroup this means passing to a subgroup of finite index which does not change the finiteness length. Reducing to a semi-simple group scheme by splitting off a central torus does also not affect the finiteness length of Γ. Using a central isogeny [Behr68, Satz 1], we can assume that the group scheme is simply connected still without any change of the finiteness length. A simply connected semi-simple group is the direct product of its almost simple factors (which remain simply connected). The finiteness length of a direct product is the minimum of the finiteness lengths of its factors. Finally, by restriction of scalars, one may assume that each factor of the direct product is absolutely almost simple (and still simply connected); see, e.g., [Kne65, Hilfssatz 7.4 and 7.5]. Kneser treats the number field case, but his arguments can be extended without difficulty. Finally, the Rank Theorem applies to the absolut almost simple factors.

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1 Behr-Harder reduction theory

In this section, we collect the results of reduction theory. Let $\mathcal{G}$ be a connected, reductive, non-commutative K-isotropic K-group. In particular, it has proper K-parabolic subgroups. In Section 12, we shall state and prove results in the slightly more general setting of reductive but not necessarily isotropic $\mathcal{G}$.

The euclidean building $X := \prod_{p \in S} X_p$ associated to $\mathcal{G}(\mathcal{O}_S)$ is a CAT(0)-space [AbBr08, Theorem 11.16]. Its visual boundary consists of parallelism classes of geodesic rays $\rho : \mathbb{R}_{\geq 0} \to X$. To each such ray, one associates a Busemann function

$$\begin{array}{rcl} \beta : X & \longrightarrow & \mathbb{R} \\ x & \mapsto & \lim\limits_{t \to \infty} (\operatorname{dist}(\rho(0), \rho(t)) - \operatorname{dist}(x, \rho(t))). \end{array}$$

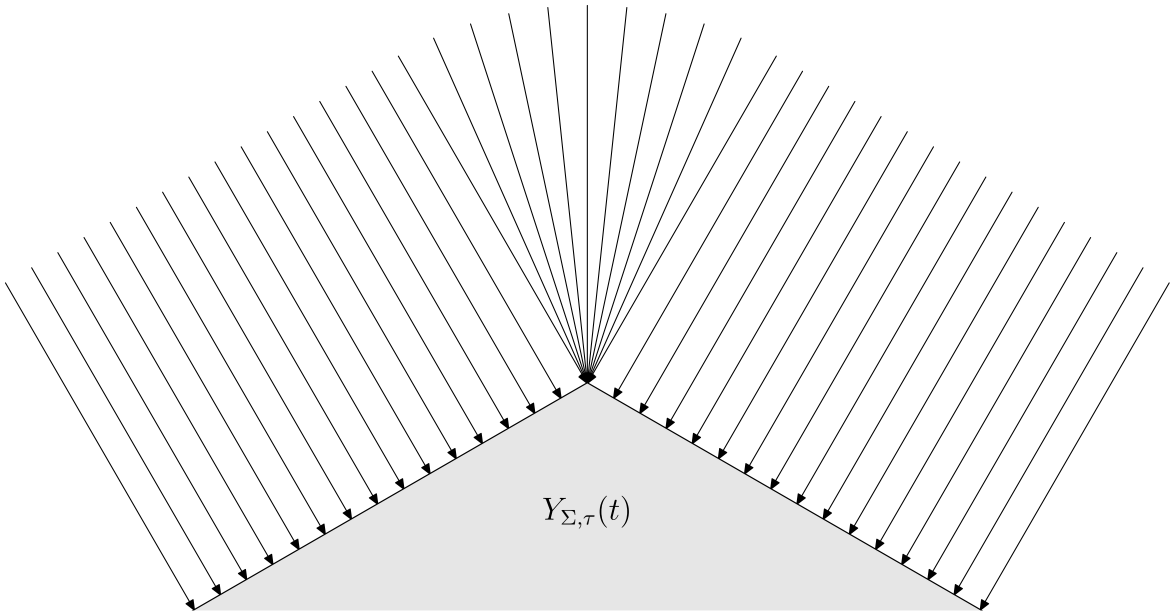

Figure 1: the closest point projection $\mathrm{pr}^t_{\Sigma,\tau}$

Two rays are parallel if and only if their corresponding Busemann functions differ by an additive constant. In particular, a Busemann function determines a unique point in the visual boundary, its <u>center</u>. See, e.g., [BrHa99, pages 267ff, in particular 8.20].

Let $\Delta := \Delta_K$ be the spherical building for the group $\mathcal{G}(K)$. It is an irreducible building whose chambers correspond to the minimal K-parabolic subgroups and whose vertices correspond to the maximal K-parabolic subgroups of $\mathcal{G}$. Let $\mathcal{C}(\Delta)$ denote its set of chambers and $\mathcal{V}(\Delta)$ its set of vertices.

The visual boundary $\partial(X)$ is the spherical join of the boundaries $\partial(X_p)$ where the join is taken over all $p \in S$. In Proposition 12.2, we describe how one can construct a Γ-invariant isometric embedding $\Delta \hookrightarrow \partial(X)$. Using this embedding, we identify vertices of Δ with points in the visual boundary $\partial(X)$. It turns out that Busemann functions centered at vertices of $\Delta \subset \partial(X)$ are not constant on any of the factors X_p of X.

Now, fix a family $(\beta_v : X \to \mathbb{R})_{v \in \mathcal{V}(\Delta)}$ of Busemann functions so that each β_v is centered at v. For any simplex τ of Δ, put $\beta_\tau(x) := \max_{v\in\tau}(\beta_v(x))$. For an apartment Σ of X and a simplex τ of Δ contained in the visual boundary $\partial(\Sigma)$, we consider the convex cones

$$Y_{\Sigma,\tau}(t) := \{x \in \Sigma \mid \beta_\tau(x) \leq t\}$$

as dependent on a real parameter t. Let

$$\mathrm{pr}^t_{\Sigma,\tau} : \Sigma \longrightarrow Y_{\Sigma,\tau}(t)$$

denote the closest point projection.

Observation 1.1. *Any two apartments Σ and Σ' containing $x \in X$ and with τ in their visual boundary can be identified via an isometry that commutes with the*

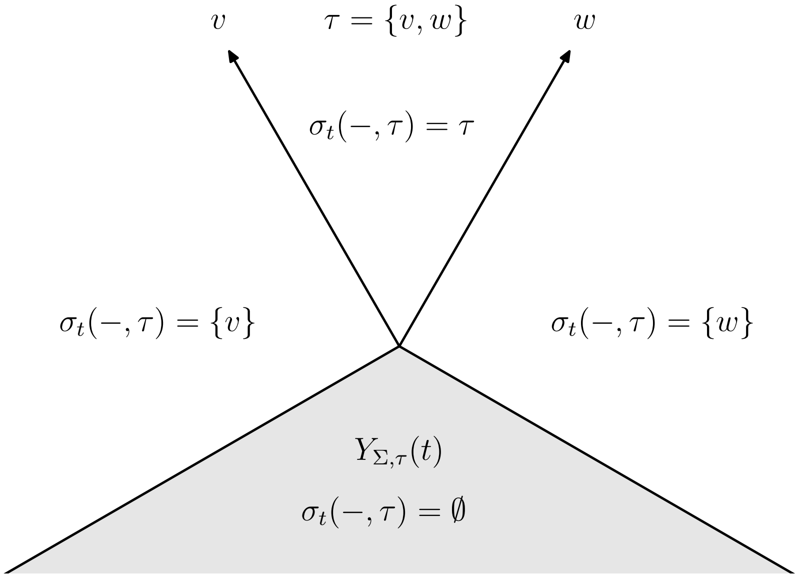

Figure 2: decomposition of an apartment

Shown is an apartment whose visual boundary contains the simplex τ and the decomposition of the apartment into regions according to the values of $\sigma_t(-,\tau)$. Note in particular how moving up the tip will affect the picture.

Busemann functions β_v for all $v \in \tau$. More precisely, there is an isomorphism of Coxeter complexes

$$\iota : \Sigma \longrightarrow \Sigma'$$

such that the following diagram commutes:

$$\begin{tikzcd} \Sigma \arrow[rr, "\iota"] \arrow[dr, "\beta_v"'] & & \Sigma' \arrow[dl, "\beta_v"] \\ & \mathbb{R} & \end{tikzcd}$$

In particular, ι identifies $Y_{\Sigma,\tau}(t)$ and $Y_{\Sigma',\tau}(t)$. Moreover,

$$\mathrm{pr}^t_{\Sigma',\tau} \circ \iota = \iota \circ \mathrm{pr}^t_{\Sigma,\tau}$$

and the values $b^t_{\tau,v}(x) := \beta_v\big(\mathrm{pr}^t_{\Sigma,\tau}(x)\big)$ are independent of the apartment Σ. □

We put

$$\sigma_t(x,\tau) := \left\{ v \in \tau \,\middle|\, b^t_{\tau,v}(x) = t \right\}.$$

Thinking within a given apartment Σ containing x and τ, the set $\sigma_t(x,\tau)$ collects precisely those vertices $v \in \tau$ whose associated inequalities $\beta_v(-) \le t$ are sharp at the point $\mathrm{pr}^t_{\Sigma,\tau}(x)$. Hence, we may delete the other inequalities:

Observation 1.2. *For any subsimplex $\sigma \subseteq \tau$ containing $\sigma_t(x,\tau)$, the closest point to x in $Y_{\Sigma,\tau}(t)$ is also the closest point to x in $Y_{\Sigma,\sigma}(t)$, i.e., $\mathrm{pr}^t_{\Sigma,\tau}(x) = \mathrm{pr}^t_{\Sigma,\sigma}(x)$. In particular, it follows that $\sigma_t(x,\tau) = \sigma_t(x,\sigma)$.* □

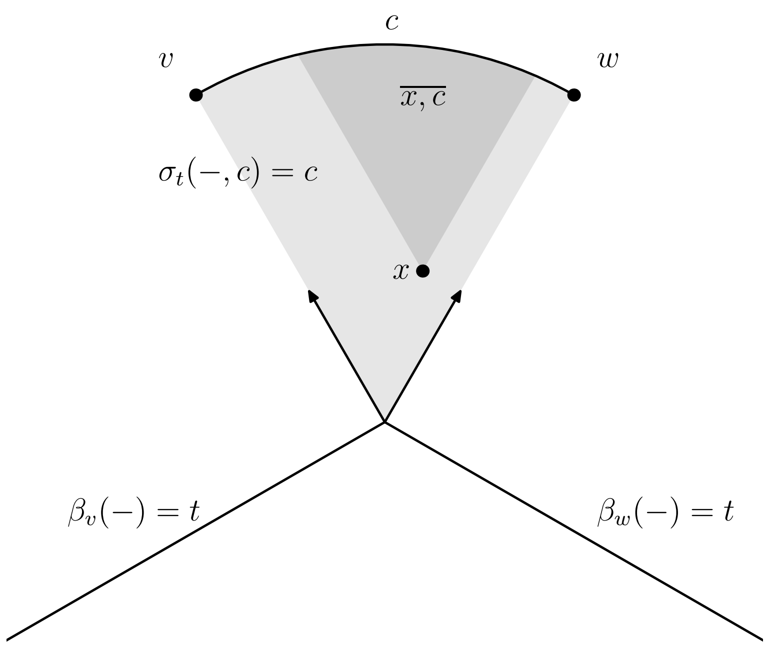

Figure 3: reducing convex hulls

The lightly shaded area is the region of points in the depicted apartment reduced by c. The darker region is the cone $\overline{x,c}$.

We say that a chamber $c \in \mathcal{C}(\Delta)$ t-reduces $x \in X$ if $\sigma_t(x,c) = c$.

Observation 1.3. *For $t \leq t'$,*

$$\sigma_{t'}(x,\tau) \subseteq \sigma_t(x,\tau) \subseteq \tau. \qquad \square$$

Corollary 1.4. *Assume $t' := \beta_\tau(x) \geq t$. Then each vertex v with $\beta_v(x) = t'$ belongs to $\sigma_t(x,\tau)$. In particular, $\sigma_t(x,\tau) \neq \emptyset$ and $\beta_\tau(x) = \beta_{\sigma_t(x,\tau)}(x)$.*

Proof. We have $\{v \in \tau \mid \beta_v(x) = t'\} = \sigma_{t'}(x,\tau) \subseteq \sigma_t(x,\tau)$. $\square$

Observation 1.5. *Assume that c t-reduces x. As illustrated in Figure 3, the chamber c t-reduces every point in the sector $\overline{x,c}$.* $\square$

A reduction datum consists of a family $(\beta_v : X \to \mathbb{R})_{v \in \mathcal{V}(\Delta)}$ of Busemann functions on the euclidean building X and two constants $r < R$ so that the following holds:

> For any chamber c that r-reduces x, the simplex $\sigma_R(x,c)$ is contained in *any* chamber c' that r-reduces x.

Observation 1.6. *If $((\beta_*), r, R_0)$ is a reduction datum and $R_1 > R_0$, then so is $((\beta_*), r, R_1)$.* $\square$

We now fix a reduction datum and say c reduces the point x if it r-reduces the point.

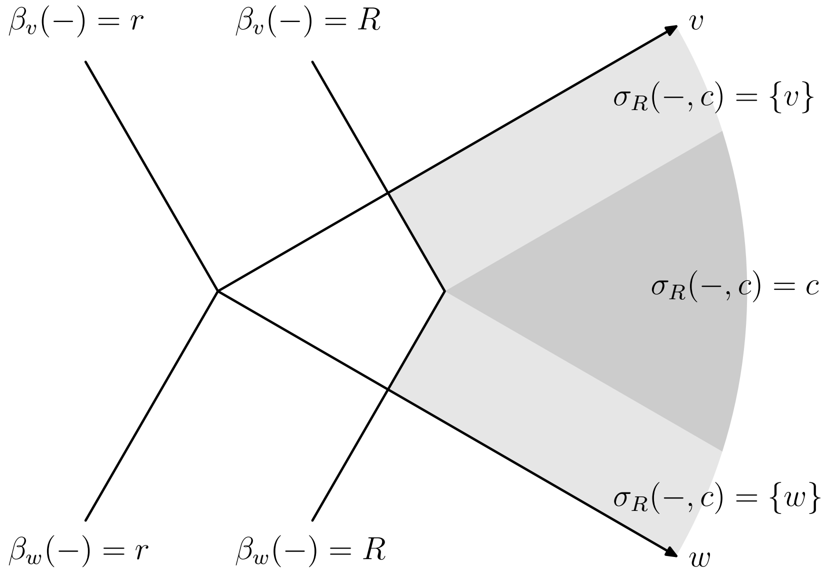

Figure 4: the upper reduction bound

Shown is the partition of the reduced region as induced by the upper reduction bound R. The chamber is $c = \{v, w\}$.

Observation 1.7. *If the chambers c and c' both reduce the point $x \in X$, then*

$$\sigma_t(x, c) = \sigma_t(x, c') \ .$$

for any $t \geq R$. Also, if $\beta_c(x) \geq R$, then $\beta_c(x) = \beta_{c'}(x)$ and $\sigma_t(x, c) \neq \emptyset$.

Proof. First, $\sigma_t(x, c) \subseteq \tau := c \cap c'$. Therefore Observation 1.2 yields $\sigma_t(x, c) = \sigma_t(x, \tau) = \sigma_t(x, c')$.

Furthermore, if $\beta_c(x) \geq R$, Corollary 1.4 implies $\beta_c(x) = \beta_{\sigma_t(x,c)}(x) = \beta_{\sigma_t(x,c')}(x) = \beta_{c'}(x)$. □

Corollary 1.8. *For any two chambers $c, c' \in \mathcal{C}(\Delta)$ both reducing the point $x \in X$ and any bound $t \geq R$, we have*

$$\beta_c(x) \leq t \qquad \textit{if and only if} \qquad \beta_{c'}(x) \leq t. \qquad \square$$

We define $\sigma_t(x) := \bigcap_{c \text{ reduces } x} \sigma_t(x, c)$. We call the (possibly empty) simplex

$$\sigma(x) := \sigma_R(x)$$

<u>close to</u> x. It is non-empty if $\beta_c(x) \geq R$ for some (and hence any) chamber c that reduces x. In this case, it equals $\sigma_R(x, c)$ for any reducing chamber.

A reduction datum is <u>Γ-invariant</u> if for each $\gamma \in \Gamma$, each vertex $v \in \mathcal{V}(\Delta)$, and each point $x \in X$, we have $\beta_{\gamma v}(\gamma x) = \beta_v(x)$. A Γ-invariant reduction datum is called <u>Γ-cocompact</u> if for each $t \geq R$, the set

$$\begin{aligned} Y_t &:= \{x \in X \mid \beta_c(x) \leq t \text{ for all } c \in \mathcal{C}(\Delta) \text{ reducing } x\} \\ &= \{x \in X \mid \beta_c(x) \leq t \text{ for some } c \in \mathcal{C}(\Delta) \text{ reducing } x\} \end{aligned}$$

has compact quotient modulo the action of Γ.

We say that a subset $B \subseteq X$ can be uniformly reduced if there is a chamber $c \in \mathcal{C}(\Delta)$ reducing simultaneously all points in B. Let d be a non-negative real number. We say that a reduction datum is d-uniform if every subset $B \subset X$ of diameter at most d can be uniformly reduced.

With these notions, we can rephrase the main theorems of Behr-Harder reduction theory as follows:

Theorem 1.9. *For every diameter d, there is a d-uniform, Γ-invariant reduction datum. It is Γ-cocompact provided $\mathcal{G}$ is absolutely almost simple.*

We give a proof in Section 12. This rendering of the statement is loosely inspired by [Behr98] where a slightly different version of the sets Y_t was used as a filtration of X: just let t tend to ∞. Ultimately, we will choose d large, although for our immediate needs, we shall only require that closed chambers in X can be uniformly reduced.

2 A blueprint for the main argument

The proof of finiteness properties for the group Γ centers around a Γ-invariant Morse function $h : X \to \mathbb{R}$ with Γ-cocompact sublevel sets and highly connected descending links. Here, we use the term "Morse function" loosely: the key feature is that directions can be ascending or descending and that there is a gradient, which is the direction of steepest ascent.

In this section, we shall construct an approximation $\hat{h} : X \to \mathbb{R}$ that almost suffices: $\hat{h}$ is Γ-invariant, Γ-cocompact and *generically* has highly connected descending links. In the following sections, we will perturb $\hat{h}$ so as to make descending links highly connected everywhere.

For a point $x \in X$, an apartment Σ containing x, and a chamber $c \in \mathcal{C}(\Delta)$ in the visual boundary $\partial(\Sigma)$ and reducing x, we define $x_{\Sigma,c} := \mathrm{pr}^R_{\Sigma,c}(x)$ to be the point in $Y_{\Sigma,c}(R)$ closest to x. We let $\hat{h}_{\Sigma,c}(x)$ be the euclidean distance from $x_{\Sigma,c}$ to x.

Proposition 2.1. *With x, Σ, and c as above, $x \in Y_R$ if and only if $x \in Y_{\Sigma,c}(R)$. The point $x_{\Sigma,c}$ is also the closest point of $Y_{\Sigma,\sigma(x)}(R)$ to x.*

If $x \notin Y_{\Sigma,c}(R)$, the straight ray $\overrightarrow{x_{\Sigma,c}, x}$ from $x_{\Sigma,c}$ through x within the apartment Σ meets $\partial(\Sigma)$ in $\sigma(x)$.

In the latter case, we denote the visual end-point of $\overrightarrow{x_{\Sigma,c}, x}$ by $\hat{e}_{\Sigma,c}(x) \in \sigma(x)$.

Proof. One could be tempted to argue $Y_{\Sigma,c}(R) = Y_R \cap \Sigma$, but that is generally not true. However, a point x reduced by c belongs to either side or neither side. By Corollary 1.8,

$$\begin{aligned} & x \in Y_{\Sigma,c}(R) \\ \iff & \beta_c(x) \leq R \\ \iff & \beta_{c'}(x) \leq R \text{ for all } c' \text{ reducing } x \\ \iff & x \in Y_R. \end{aligned}$$

By Observation 1.2, $x_{\Sigma,c} = \mathrm{pr}_{\Sigma,\sigma_R(x,c)}(x)$, i.e., $x_{\Sigma,c}$ is the point of $Y_{\Sigma,\sigma(x)}(R)$ closest to x. If $x \notin Y_{\Sigma,c}(R)$ then $x \neq x_{\Sigma,c}$ and, in the apartment Σ, we can draw the ray $[x_{\Sigma,c}, x)$. This ray lies in the normal cone to $x_{\Sigma,c}$ of the convex body $Y_{\Sigma,\sigma(x)}(R)$. This normal cone is spanned by the gradients of the Busemann functions β_v for $v \in \sigma(x)$. Hence $[x_{\Sigma,c}, x)$ meets $\partial(\Sigma)$ in $\sigma(x)$. □

Now, we can see that $\hat{h}_{\Sigma,c}(x)$ and $\hat{e}_{\Sigma,c}(x)$ are independent of the choices of Σ and c.

Corollary 2.2. *If Σ' and c' is another pair with $c' \subset \partial(\Sigma')$, $x \in \Sigma'$, and c' reducing x, then $\hat{h}_{\Sigma,c}(x) = \hat{h}_{\Sigma',c'}(x)$ and $\hat{e}_{\Sigma,c}(x) = \hat{e}_{\Sigma',c'}(x)$.*

Proof. First, we have the following chain of equivalences:

$$
\begin{aligned}
& \hat{h}_{\Sigma,c}(x) = 0 \\
\iff & x \in Y_{\Sigma,c}(R) \\
\iff & x \in Y_R \\
\iff & x \in Y_{\Sigma',c'} \\
\iff & \hat{h}_{\Sigma',c'}(x) = 0
\end{aligned}
$$

Hence, whether or not $\hat{h}_{\Sigma,c}(x) = 0$ does depend neither on Σ nor c.

Both chambers, c and c', reduce x. Hence, $\sigma(x) \subseteq c \cap c'$ and the convex hull $\overline{x, \sigma(x)}$ of x and $\sigma(x)$ lies in $\Sigma \cap \Sigma'$. Hence, there is an isometry $\iota : \Sigma \to \Sigma'$ fixing $\overline{x, \sigma(x)}$ pointwise. Hence for each $v \in \sigma(x) \subseteq c \cap c'$, the following diagram commutes:

$$
\begin{tikzcd}
\Sigma \arrow[rr, "\iota"] \arrow[dr, "\beta_v"'] & & \Sigma' \arrow[dl, "\beta_v"] \\
& \mathbb{R} &
\end{tikzcd}
$$

Hence, ι identifies $Y_{\Sigma,\sigma(x)}$ with $Y_{\Sigma',\sigma(x)}$ and $x_{\Sigma,c}$ with $x_{\Sigma',c'}$. It follows that $\hat{e}_{\Sigma,c}(x) = \hat{e}_{\Sigma',c'}(x)$ and $\hat{h}_{\Sigma,c}(x) = \hat{h}_{\Sigma',c'}(x)$. □

Thus, we define $\hat{h}(x) := \hat{h}_{\Sigma,c}(x)$ and $\hat{e}(x) := \hat{e}_{\Sigma,c}(x)$ where c is a chamber reducing x and Σ is an apartment containing x and whose boundary contains c.

The properties of $\hat{h}$ that we are about to establish roughly say that $\hat{h}$ qualifies as a Morse function for the analysis of Γ.

Observation 2.3. *The function $\hat{h}$ is defined entirely in terms of a Γ-invariant reduction datum, hence it is itself Γ-invariant. I.e., $\hat{h}(\gamma x) = \hat{h}(x)$ for any $\gamma \in \Gamma$ and any point $x \in X$.* □

Proposition 2.4. *Each sublevel set $\hat{h}^{-1}([0, s])$ has compact quotient modulo the action of Γ.*

Proof. We use the Γ-cocompactness of the reduction datum. More precisely, we show that for any $s \geq 0$ the sublevel set $\hat{h}^{-1}([0, s])$ is contained in the Γ-cocompact set Y_{s+R}. It is easier to prove the contrapositive. So assume that we have a point $x \notin Y_{s+R}$. We have to argue that $\hat{h}(x) > s$.

As $x \notin Y_{s+R}$ there is a chamber c reducing x with $\beta_c(x) > s + R$, i.e., there is a vertex $v \in c$ with $\beta_v(x) > s + R$. To estimate $\hat{h}(x)$ we additionally choose an apartment Σ containing x that has c in its visual boundary. Then

$$\begin{aligned} \hat{h}(x) &= \hat{h}_{\Sigma,c}(x) \\ &= \operatorname{dist}(x_{\Sigma,c}, x) \\ &\geq \beta_v(x) - \beta_v(x_{\Sigma,c}) \\ &> (s + R) - R = s. \end{aligned}$$

This completes the proof. The first inequality follows in general from the definition of Busemann functions and the triangle inequality. However, as Σ is a euclidean space, we have a more precise statement: $\beta_v(x) - \beta_v(x_{\Sigma,c}) = \cos(\vartheta) \operatorname{dist}(x_{\Sigma,c}, x)$ where ϑ is the angle between the ray $[x_{\Sigma,c}, x)$ and the gradient of β_v. $\square$

Proposition 2.5. *The function $\hat{h} : X \to \mathbb{R}$ is continuous.*

Proof. Here, we use the hypothesis that the chosen reduction datum is sufficiently uniform. More precisely, we shall use that every chamber in X can be uniformly reduced. So let C be a chamber in X, let $c \in \mathcal{C}(\Delta)$ be a chamber uniformly reducing C, and let Σ be an apartment containing C whose visual boundary contains c. Then, the functions $\hat{h}$ and $\hat{h}_{\Sigma,c}$ agree on C. In particular, $\hat{h}$ restricts to a continuous function on C. As the euclidean building X carries the weak topology with respect to its chambers, the function $\hat{h}$ is continuous. $\square$

Morse functions are supposed to be differentiable and not merely continuous. The following statements make precise the sense in which $\hat{h}$ induces a gradient field and what should be considered flow lines of this gradient field. First we treat the gradient, i.e., we describe the behavior of $\hat{h}$ on small scales.

Proposition 2.6. *Let x and y be two points in X that lie within a common closed chamber. Then*

$$\hat{h}(y) - \hat{h}(x) \leq \operatorname{dist}(y, x)$$

with equality if and only if y lies on the ray $[x, \hat{e}(x))$. In case of equality, moreover $\hat{e}(x) = \hat{e}(y)$.

Thus, we define the gradient of $\hat{h}$ at the point x to be the direction $\nabla_x \hat{h} \in \operatorname{lk}(x)$ defined by the ray $[x, \hat{e}(x))$. It is the unique direction of fastest ascent, and $\hat{h}$ grows in that direction with unit speed.

Proof. Choose c and Σ so that $\hat{h}$ and $\hat{h}_{\Sigma,c}$ agree on the segment $[x, y]$. Then

$$\begin{aligned}
\hat{h}(y) - \hat{h}(x) &= \hat{h}_{\Sigma,c}(y) - \hat{h}_{\Sigma,c}(x) \\
&= \operatorname{dist}(y_{\Sigma,c}, y) - \operatorname{dist}(x_{\Sigma,c}, x) \\
&\leq \operatorname{dist}(x_{\Sigma,c}, y) - \operatorname{dist}(x_{\Sigma,c}, x) \\
&\leq \operatorname{dist}(y, x).
\end{aligned}$$

We have equality in the last step if and only if x lies on the straight line segment $[x_{\Sigma,c}, y]$. In this case, however, the segment $[x_{\Sigma,c}, y]$ is normal to $Y_{\Sigma,c}(R)$. Therefore, $y_{\Sigma,c} = x_{\Sigma,c}$ whence we have equality throughout and $\hat{e}(x) = \hat{e}(y)$. □

Observation 2.7. *Every line segment in X cuts through finitely many chambers. Hence, one can easily "integrate" the local information provided by the previous proposition: for any two points y and x, the geodesic segment $[x, y]$ can be subdivided into finitely many segments each of which is supported by a closed chamber. That is, there are points*

$$x = x_0, x_1, \ldots, x_n = y \in [x, y]$$

such that $[x, y]$ is the concatenation of the segments $[x_i, x_{i+1}]$. Applying the previous proposition to each of those segments, we obtain

$$\hat{h}(y) \leq \hat{h}(x) + \operatorname{dist}(y, x)$$

with equality if and only if $y \in [x, \hat{e}(x))$. In the case of equality, $\hat{e}(y) = \hat{e}(x)$. □

Thus, we can regard the unit speed geodesic ray from x to $\hat{e}(x)$ as a flow line of the gradient field.

Now we are ready to give a first, albeit failing, attempt to prove the Rank Theorem. The actual proof given below follows the same lines, and this argument will help identify exactly the shortcomings of $\hat{h}$ that we have to address in the following sections.

The S-arithmetic group Γ acts on the euclidean building X, which is a CAT(0)-space and therefore contractible. Cell stabilizers of this action are finite as X is a proper CAT(0)-space and Γ is discrete in its isometry group. For each positive real s let $X(s)$ be the largest subcomplex of X fully contained in the sublevel set $\hat{h}^{-1}([0, s])$. By Proposition 2.4, the orbit space of $X(s)$ modulo the action of Γ is compact. Should $X(s)$ be $(d-2)$-connected for some s, then [Bro87, Propositions 1.1 and 3.1] would imply that Γ is of type F_{d-1}.

The aim of combinatorial Morse theory is to describe how the homotopy type of sublevel complexes $X(s)$ change as s varies. This description should be in terms of purely local information about the function $\hat{h}$. Crucial are descending links, i.e., the set of directions at a given point x along which the function $\hat{h}$ decreases. More precisely, a cell τ containing the vertex x is considered descending if $\hat{h}$ assumes its maximum on τ at and only at x. The descending cells at x form the descending link at x. If all vertices of x with $s \leq \hat{h}(x) \leq s'$ have m-connected descending links, the inclusion $X(s) \subseteq X(s')$ of sublevel complexes induces isomorphisms in homotopy groups π_i for all $i \leq m$. As the euclidean building X is contractible, these

isomorphisms in the π_i imply that already some $X(s)$ is m-connected. Thus, using Brown's criterion from above, we are reduced to the question of whether descending links are $(d-2)$-connected.

In smooth Morse theory, the descending link is an infinitesimal notion and a direction is descending if it spans an obtuse angle with the gradient. In the combinatorial setting, whether an edge determines an ascending or descending direction in the link of an adjacent vertex depends on the values of the Morse function at the end points. Hence, the descending link is a local rather than an infinitesimal notion. Often, however, the infinitesimal behavior of the Morse function is good enough an approximation: at a *generic* vertex x an adjacent edge is descending if and only if it spans an obtuse angle with the gradient $\nabla_x \hat{h}$. At those vertices where predictions based solely on the gradient are correct, descending links are therefore hemisphere complexes, whose connectivity properties are given in [Schu10].

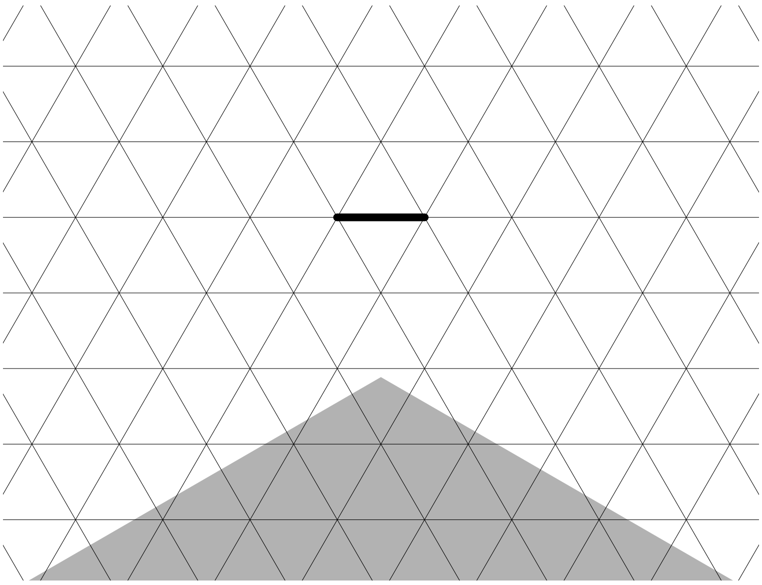

Figure 5: a bad edges

The shaded area in the bottom is $Y_{\Sigma,c}(R)$. The "corridor" above the tip yields an infinite family of "bad edges".

This strategy almost succeeds. Unfortunately, the gradient criterion is sometimes wrong. Figure 5 shows an apartment Σ with the convex set $Y_{\Sigma,c}(R)$ drawn in. Suppose, c reduces the marked edge. Then, the edge spans an obtuse angle with the gradient at either end point. Hence, the gradient criteria for both vertices are in direct conflict with one another. The reason is that the gradient criterion only makes correct predictions on an infinitesimal scale. The Morse function $\hat{h}$ actually decreases along the edge from either end toward the center. Beyond the center point, however, $\hat{h}$ increases again, spoiling the prediction based on the gradient. The picture 5 also shows that this problem occurs "arbitrarily far out", i.e., we cannot avoid it by considering $X(s)$ for some high value of s: since the reduction datum is geometric, the chamber c also reduces all the edges in the "corridor" above the marked edge whence

$\hat{h}$ along these edges can be read off in the picture.

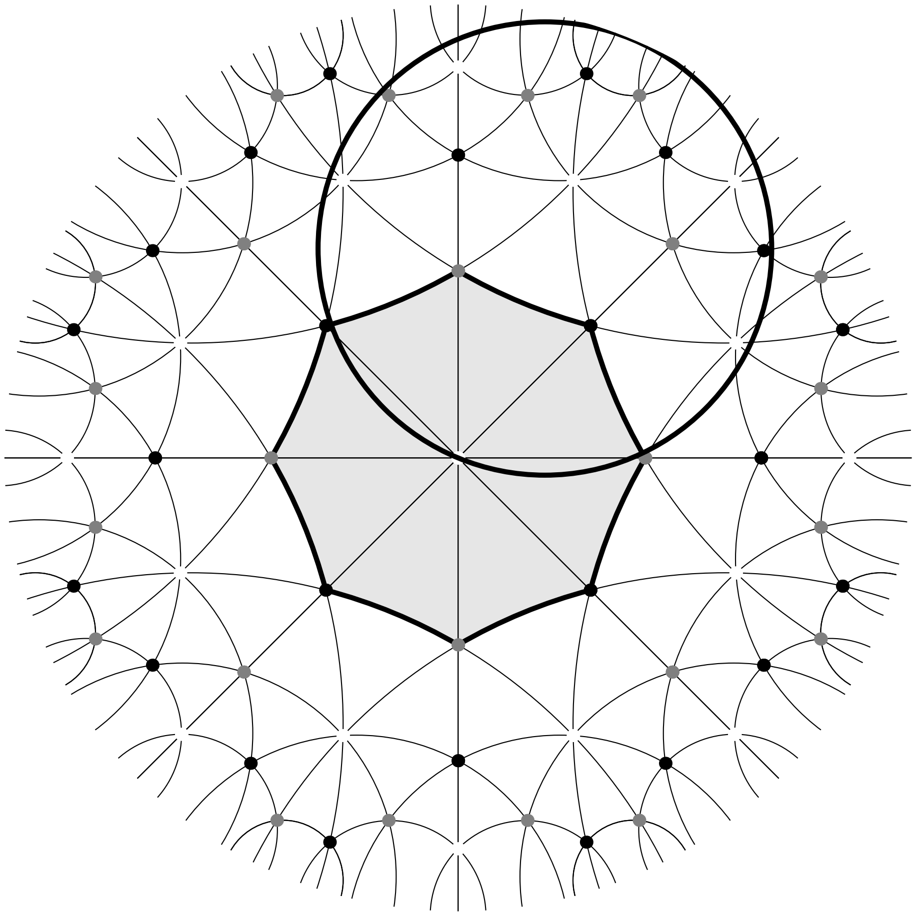

Figure 6: a star in a hyperbolic Coxeter complex

We note that the predictive powers of the gradient hinge upon the underlying apartments being *euclidean* spaces. A Coxeter complex in hyperbolic space does not scale: its edges have lengths determined by the Coxeter diagram. If vertices are too far apart, the infinitesimal nature of the gradient renders it useless even for predicting the value of a Busemann function on neighbors. This phenomenon is shown in Figure 6. In particular, one cannot expect descending links to be hemisphere complexes. This matches examples of Abramenko in the compact hyperbolic case where the finiteness length of a lattice falls short of the dimension of the building on which the groups acts naturally.

Our main task will be to alter the Morse function $\hat{h}$ to make gradients consistent, i.e., we do not want to see edges that span obtuse angles with gradient vectors at either end. The obstruction is, of course, that we want to keep high connectivity of descending links at vertices where they are already fine.

The remainder of this paper is organized as follows. After some preliminaries on zonotopes in Section 3 and spherical buildings in Section 4, we define, in Section 5, a primary Morse function h (the <u>height</u>), which is a perturbation of $\hat{h}$ discussed

above. It is Γ-invariant, Γ-cocompact, continuous, and induces a gradient field with geodesic flow lines. It improves upon $\hat{h}$ in that the gradient criterion never leads to inconsistencies. However, we cannot avoid <u>h-flat</u> cells, e.g., edges on which h is constant. To break ties, i.e., to determine which vertex of such an edge to add first in filtering the euclidean building, we introduce a secondary and even a tertiary Morse function in Section 7. Here we rely on the notion of <u>depth</u> introduced in [BW08] and further developed in [Witz10]. The analysis of descending links for h is carried out in Section 9 whereas Section 10 derives the Rank Theorem. The final three sections are devoted to reduction theory.

3 A small convex geometry toolkit

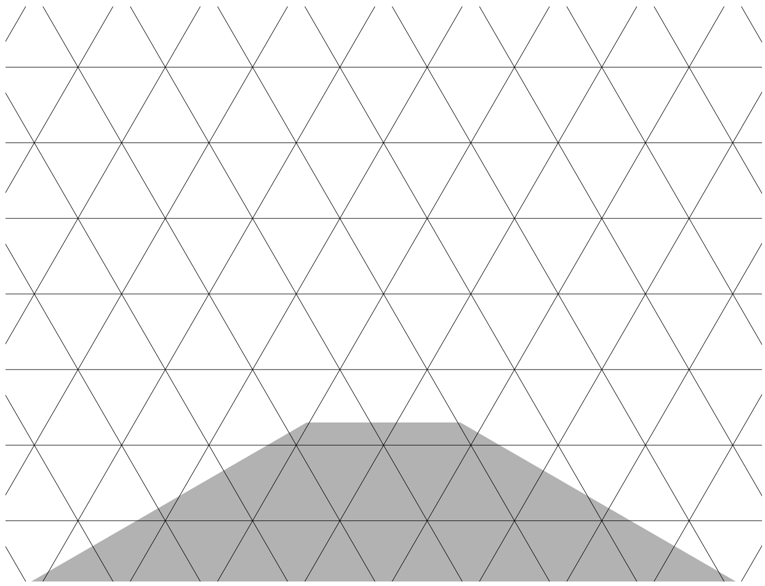

Figure 7: flattening the tip

Flattening the tip removes corridors of bad edges.

We address the problem illustrated in Figure 5 by changing the shape of $Y_{\Sigma,c}(R)$. We will flatten the tip as shown in Figure 7. Then the gradient field becomes consistent along edges.

Let $\mathbb{E}$ denote a euclidean space with inner product $\langle -, - \rangle$ and origin $\mathbf{0}$. Let F be the face of some convex polytope P. The <u>normal cone</u>

$$N(F) := \left\{ \mathbf{n} \in \mathbb{E} \;\middle|\; \langle \mathbf{n}, \mathbf{z} \rangle = \max_{\mathbf{z}' \in P} \langle \mathbf{n}, \mathbf{z}' \rangle \text{ for all } \mathbf{z} \in F \right\}$$

is the set of all $\mathbf{n} \in \mathbb{E}$ such that the function $\langle \mathbf{n}, - \rangle$ restricted to P assumes its maximum on the points in F. It is a closed convex cone. For any point $\mathbf{x} \in \mathbb{E}$, the closest point projection onto P satisfies:

$$\mathrm{pr}_P(\mathbf{x}) \in F \qquad \text{if and only if} \qquad \mathbf{x} - \mathrm{pr}_P(\mathbf{x}) \in N(F)$$

For a finite subset $D \subset \mathbb{E}$, the compact convex polytope

$$Z(D) := \left\{ \sum_{\mathbf{d} \in D} a_{\mathbf{d}} \mathbf{d} \;\middle|\; 0 \leq a_{\mathbf{d}} \leq 1 \text{ for all } \mathbf{d} \in D \right\}$$

is called the zonotope spanned by D. This construction ensures:

Observation 3.1. *Through every point $\mathbf{z} \in Z(D)$ and every $\mathbf{d} \in D$, there is a line segment parallel to $[\mathbf{0}, \mathbf{d}]$ inside $Z(D)$.* □

Let $P \subset \mathbb{E}$ be a compact convex polytope. We call $D \subset \mathbb{E}$ saturated with respect to P, if for any two vertices $\mathbf{v}, \mathbf{v}' \in P$, at least one of $\mathbf{v} - \mathbf{v}'$ or $\mathbf{v}' - \mathbf{v}$ lies in D.

Proposition 3.2. *Let $D \subset \mathbb{E}$ be saturated with respect to the compact convex polytope $P \subset \mathbb{E}$. Then any translate of $Z(D)$ that intersects P contains a vertex of P. Equivalently: whenever one translates P to intersect $Z(D)$ one has to move a vertex of P into $Z(D)$.*

Proof. Since differences of vertices are invariant under translation, we may also assume that P intersects $Z(D)$. Let $\mathbf{z}$ be a point in the intersection. We have to show that $Z(D)$ contains a vertex of P.

Let $\mathbf{v}$ and $\mathbf{v}'$ be two vertices of P on opposite parallel supporting hyperplanes. Choose the labels so that $\mathbf{v} - \mathbf{v}' \in D$. Any translate of the segment $[\mathbf{v}, \mathbf{v}']$ through the point $\mathbf{z}$ meets the boundary of P, and Observation 3.1 implies that there is such a translate that stays inside $Z(D)$. Hence, P has a face F that intersects $Z(D)$. By induction on the dimension, we may therefore conclude that $Z(D)$ contains a vertex of F. □

Corollary 3.3. *If D and P are as in the proposition, i.e., D is saturated with respect to P, then $P + Z(D) = P^{(0)} + Z(D)$ where $P^{(0)}$ is the set of vertices of P.*

Proof. Let $x \in P + Z(D)$, i.e., there is a point $\mathbf{z} \in P$ with $x \in \mathbf{z} + Z(D)$. Then $\mathbf{z} \in x + Z(-D)$. By Proposition 3.2, there is a vertex $\mathbf{v} \in P^{(0)}$ with $\mathbf{v} \in x + Z(-D)$, which implies $x \in \mathbf{v} + Z(D)$. □

Let $f : \mathbb{E} \to \mathbb{R}$ be a continuous function and let C be a non-empty compact convex subset of $\mathbb{E}$. Define:

$$\begin{array}{rcl} f_C : \mathbb{E} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \min \{ f(y) \mid y \in x + C \} \end{array}$$

Observation 3.4. *If f is convex then so is f_C.* □

Proposition 3.5. *Let $f : \mathbb{E} \to \mathbb{R}$ be continuous, let $P \subset \mathbb{E}$ be a compact convex polytope, and let $D \subset \mathbb{E}$ be a finite subset containing all differences $\mathbf{x} - \mathbf{x}'$ for any two vertices $\mathbf{x}$ and $\mathbf{x}'$ of P. Then the following hold:*

1. *The min-set of $f_{Z(D)}$ on P contains a vertex of P.*

2. *If f is a convex function then the max-set of $f_{Z(D)}$ on P is a face of P.*

Proof. Part (1) follows immediately from Corollary 3.3. As for part (2), put:

$$\mathbf{V} := \left\{ \mathbf{v} \in \text{max-set}\big(f_{Z(D)}|_P\big) \;\middle|\; \mathbf{v} \text{ is vertex of } P \right\}$$

Since $f_{Z(D)}$ is a convex function, $\mathbf{V}$ is not empty. Let F be the smallest face of P containing $\mathbf{V}$, and let $\overline{\mathbf{V}}$ be the convex hull of $\mathbf{V}$. By part (a), the function $f_{Z(D)}|_{\overline{\mathbf{V}}}$ assumes its minimum in a vertex, i.e., a point in $\mathbf{V}$. Hence f is constant on $\overline{\mathbf{V}}$, as $\mathbf{V}$ consists of points in P where $f_{Z(D)}$ is maximal. As F is the smallest face of P containing $\overline{\mathbf{V}}$ and $f_{Z(D)}$ is a convex function, $f_{Z(D)}$ is constant on F. □

Example 3.6. Let $Y \subset \mathbb{E}$ be a non-empty closed convex set. Consider the distance from Y as a function

$$\begin{array}{rcl} f : \mathbb{E} & \longrightarrow & \mathbb{R} \\ \mathbf{x} & \mapsto & \operatorname{dist}(\mathbf{x}, Y). \end{array}$$

Then, for any non-empty compact convex set $C \subset \mathbb{E}$ and for any $\mathbf{x} \in \mathbb{E}$, we have:

$$\begin{aligned} f_C(\mathbf{x}) &= \min_{\mathbf{y}\in C} \left(f(\mathbf{x}+\mathbf{y})\right) \\ &= \min_{\mathbf{y}\in C} \left(\operatorname{dist}(\mathbf{x}+\mathbf{y}, Y)\right) \\ &= \min_{\mathbf{y}\in C, \mathbf{z}\in Y} \left(\operatorname{dist}(\mathbf{x}+\mathbf{y}, \mathbf{z})\right) \\ &= \min_{\mathbf{y}\in C, \mathbf{z}\in Y} \left(\operatorname{dist}(\mathbf{x}, \mathbf{z}-\mathbf{y})\right) \\ &= \operatorname{dist}(\mathbf{x}, Y - C) \end{aligned}$$

Hence, the functions f_C and $\operatorname{dist}(-, Y - C)$ coincide. □

4 Some subcomplexes of spherical buildings

To deduce finiteness properties, we use the well-established technique of filtering a complex upon which the group acts. The main task, as usual, is to control the homotopy type of relative links that arise in the filtration. In this section, we collect the results concerning connectivity properties of those subcomplexes of spherical buildings that we will encounter.

Let M be euclidean or hyperbolic space or a round sphere. We call an intersection of a non-empty family of closed half-spaces (or hemispheres in the latter case) <u>demi-convex</u>. We call a subset of M <u>fat</u> if it has non-empty interior. Note that a proper open convex subset of M is contained in an open hemisphere.

Observation 4.1. *Let $A \subset M$ be fat and demi-convex and let $B \subset M$ be proper, open, and convex. If A and B intersect, then $A \setminus B$ strongly deformation retracts onto the boundary part $\partial(A) \setminus B$.*

Proof. Note that B intersects the interior of A since every boundary point of the convex set A is an accumulation point of interior points because A is fat. Choose x in the intersection. Note that A is star-like with regard to x, and the geodesic projection away from x restricts to the deformation retraction we need. □

We call a CW-complex <u>geometric</u> if its cells carry a spherical, euclidean, or hyperbolic structure in which they are demi-convex (i.e., each cell is an intersection of half-spaces in the model geometry). Also, we require attaching maps to be isometric embeddings. Iterated application of the projection trick yields:

Proposition 4.2. *Suppose that L is a geometric CW-complex. Let B be an open subset of L that intersects each cell in a convex set. Then there is a strong deformation retraction*

$$\rho_L : L \setminus B \longrightarrow L^B$$

of $L \setminus B$ onto its maximal subcomplex.

Proof. First, we assume that L has finite dimension. Let τ be a maximal cell of L. If $\tau \subseteq B$, the cell τ does not intersect $L \setminus B$ and we do not need to do anything. If τ avoids B, the map ρ must be the identity on τ. Otherwise, let x be a point in the intersection $\tau \cap B$ chosen in the relative interior of τ. Projecting away from x, as in Observation 4.1, deformation retracts $\tau \setminus B$ onto $\partial(\tau) \setminus B$. The maps constructed for two maximal cells agree on their intersection. Hence we can paste all these maps together to get a deformation retraction of $L \setminus B$ onto $L' \setminus B$ where L' is L with the interiors of all maximal cells intersecting B removed.

Now, L' has other maximal cells, which might intersect B. Using the same construction for L', we obtain another deformation retraction $L' \setminus B \to L'' \setminus B$. We keep going, removing more and more cells intersecting B. Since the dimension of L is finite, the process terminates after finitely many steps. The composition of the maps thus obtained is the strong deformation retraction from $L \setminus B$ onto L^B. This proves the claim for finite dimensional L.

Note that the construction is local: what it does on a cell is only determined by the intersection of this cell with the set B. Hence, the deformation retraction is compatible with subcomplexes. More precisely, if K is a subcomplex of L, then the deformation retractions ρ_L and ρ_K from above are constructed such that ρ_K is the restriction of ρ_L to K. It follows that the pair $(L \setminus B, K \setminus B)$ is homotopy equivalent to (L^B, K^B). Applying this observation to pairs of skeleta, the claim follows by standard arguments in the case that L has infinite dimension. □

Let Δ be a spherical building. We regard Δ as a metric space with the angular metric. So each apartment is a round sphere of radius 1. When Δ is a finite building, the topology induced by the metric agrees with the weak topology it carries as a simplicial complex. For locally infinite buildings, both topologies differ and we will use the weak topology throughout for the building and all its subcomplexes.

Proposition 4.3. *Let Δ be a spherical building and fix a chamber C in Δ. Let $B \subset \Delta$ be a subset such that, for any apartment Σ containing C the intersection $B \cap \Sigma$ is*

a proper, open, and convex subset of the sphere Σ. Then the space $Y := \Delta \setminus B$ and its maximal subcomplex Δ^B are both $(\dim(\Delta) - 1)$-connected. The complex Δ^B has dimension $\dim(\Delta)$ and hence is spherical of this dimension.

Remark 4.4. Using $B = \emptyset$ in Proposition 4.3, we obtain the Solomon-Tits Theorem as a special case. Theorem A of [Schu10], whose proof inspired the argument given below, is the special case where B is open, convex, and of diameter strictly less than π.

Proof of Proposition 4.3. We observe first that Proposition 4.2 implies that the subset Y and its maximal subcomplex Δ^B are homotopy equivalent. Therefore, it suffices to prove that Y is $(\dim(\Delta) - 1)$-connected.

We have to contract spheres of dimensions up to $\dim(\Delta) - 1$. Let $S \subseteq Y$ be such a sphere. Since S is compact in Δ, it is covered by a finite family of apartments and we can apply [v.He03, Lemma 3.5]: there is a finite sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ such that (a) each Σ_i contains C, (b) the sphere S is contained in the union $\bigcup_i \Sigma_i$, and most importantly, (c) for each $i \geq 2$ the intersection $\Sigma_i \cap (\Sigma_1 \cup \cdots \cup \Sigma_{i-1})$ is a union of closed half-apartments, each of which contains C. Put $L_i := \Sigma_1 \cup \cdots \cup \Sigma_i$ and observe that L_i is obtained from L_{i-1} by gluing in the closure $A_i := \overline{\Sigma_i \setminus (\Sigma_1 \cup \cdots \cup \Sigma_{i-1})}$ along the boundary $\partial(A_i)$ of A_i in Σ_i. Note that A_i is fat and demi-convex.

Now, we can build $L_n \setminus B$ inductively. We begin with $L_1 \setminus B$, which is contractible. The space $L_i \setminus B$ is obtained from $L_{i-1} \setminus B$ by gluing in $A_i \setminus B$ along $\partial(A_i) \setminus B$. If A_i and B are disjoint, this is a cellular extension of dimension $\dim(\Delta)$ as A_i is fat. Otherwise, Observation 4.1 implies that $A_i \setminus B$ deformation retracts onto $\partial(A_i) \setminus B$, whence $L_i \setminus B$ and $L_{i-1} \setminus B$ are homotopy equivalent in this case. In the end, the sphere S can be contracted inside $L_n \setminus B$. □

Corollary 4.5. *Let Δ be a* finite *spherical building and fix a chamber C in Δ. Let $A \subset \Delta$ be a subset such that, for any apartment Σ containing C the intersection $A \cap \Sigma$ is a closed convex subset of diameter strictly less than π in the sphere Σ. Then the space $Y := \Delta \setminus A$ and its maximal subcomplex Δ^A are both $(\dim(\Delta) - 1)$-connected. The complex Δ^A has dimension $\dim(\Delta)$ and hence is spherical of this dimension.*

Proof. The building Δ is finite, hence A is compact. Let B be an ε-neighborhood of A. Choosing ε sufficiently small, we can ensure that B satisfies the hyptheses of Proposition 4.3, that $\Delta \setminus B$ and $\Delta \setminus A$ are homotopy equivalent, and that $\Delta^B = \Delta^A$ □

An interesting special case of Proposition 4.3, also already noted in [Schu10], is obtained when B is chosen as the open $\frac{\pi}{2}$-ball around a fixed point $n \in \Delta$, which we think of as the north pole. Then the complex $\Delta^{\geq \frac{\pi}{2}}(n) := \Delta^B$ is a closed hemisphere complex and $\dim(\Delta)$-spherical by Proposition 4.3. The argument fails badly if B is chosen as the *closed ball* of radius $\frac{\pi}{2}$ around n. In fact, the open hemisphere complex $\Delta^{> \frac{\pi}{2}}(n)$ spanned by all vertices avoiding the closed ball B generally is not $\dim(\Delta)$-spherical: the dimension of $\Delta^{> \frac{\pi}{2}}(n)$ might be too small. The main result of Schulz is that this is the only obstruction.

Proposition 4.6 (see [Schu10, Theorems A and B]). *The open hemisphere complex* $\Delta^{>\frac{\pi}{2}}(n)$ *is spherical of dimension* $\dim(\Delta_{\mathrm{ver}})$*. If* Δ *is thick, then neither open nor closed hemisphere complexes in* Δ *are contractible.*

The subcomplex $\Delta_{\mathrm{ver}}(n)$ is defined as follows: The equator $\Delta^{=\frac{\pi}{2}}(n)$ is the subcomplex spanned by those points in Δ of distance $\frac{\pi}{2}$ from n. Recall that Δ decomposes as a join of unique irreducible factors. The horizontal part $\Delta_{\mathrm{hor}}(n)$ is the join of all factors fully contained in the equator. The complex $\Delta_{\mathrm{ver}}(n)$ is the join of the other irreducible factors. In particular,

$$\Delta = \Delta_{\mathrm{hor}}(n) * \Delta_{\mathrm{ver}}(n)\,. \tag{1}$$

5 Height

We now begin the proof of the Rank Theorem proper. Let $\hat{\Sigma}$ be a euclidean Coxeter complex upon which the apartments of X are modeled, and let $\mathbb{E}$ be the underlying euclidean space where the origin $\mathbf{0}$ shall correspond to a special vertex in $\hat{\Sigma}$. Let W be the spherical Weyl group generated by the walls of $\hat{\Sigma}$ through $\mathbf{0}$. For constructing zonotopes, we shall choose an admissible subset $D \subset \mathbb{E}$, i.e., we require that D be finite, W-invariant, and symmetric with respect to the origin $\mathbf{0}$. In the course of the argument, we will need to strengthen the requirements on D, but we begin with any admissible D. Since D is invariant with respect to the maximal Weyl group, the subset $x + Z(D)$ is well-defined in any apartment Σ of X containing x.

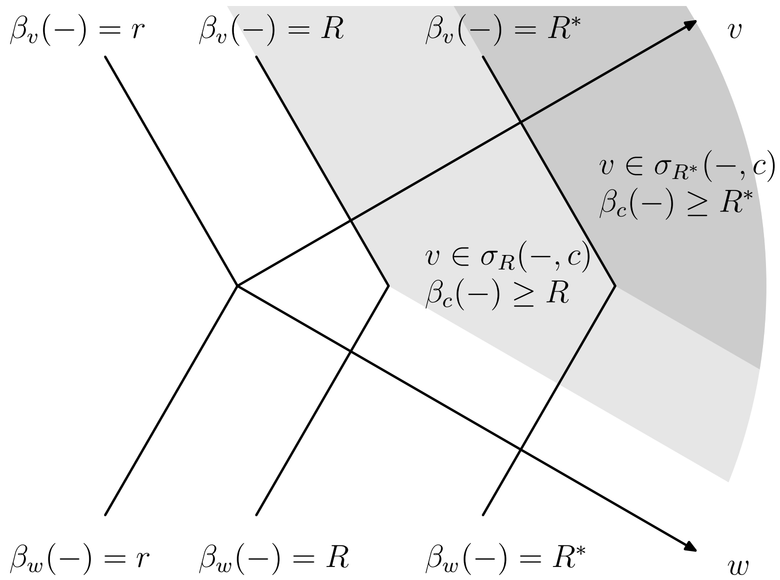

Figure 8: moving out the tip

Given a fixed distance bound L, we can choose R^* large enough so that $\beta_c(x) \geq R^*$ and $v \in \sigma_{R^*}(x, c)$ implies $v \in \sigma_R(y, c)$ for any two points x and y of distance at most L: every reduced point L-close to the darker area still lies within the lightly shaded region.

Observation 5.1. *Given a reduction datum $((\beta_*), r, R)$ for an absolutely almost simple group, there exists a bound $R^* \geq R$ such that the following implication holds:*

$$\beta_c(x) \geq R^* \text{ and } y \in x + Z(D) \qquad \Longrightarrow \qquad \sigma_{R^*}(y, c) \subseteq \sigma_R(x, c)$$

Here, $c \in \mathcal{C}(\Delta)$ is any chamber and $x, y \in X$ are any two points that lie in a common apartment whose visual boundary contains c.

See Figure 8 for a picture that explains how to choose R^* for a single vertex. Since there are only finitely many types of vertices in the building Δ, one can choose R^* large enough for every type. □

We modify the construction given in Section 2. Let $((\beta_*), r, R)$ be a reduction datum that reduces chambers in X uniformly. Fix R^* as in Observation 5.1. For a point $x \in X$, an apartment Σ containing x, and a chamber $c \in \mathcal{C}(\Delta)$ in the visual boundary $\partial(\Sigma)$ and reducing x, we define

$$h_{\Sigma,c}(x) := \operatorname{dist}(x + Z(D), Y_{\Sigma,c}(R^*)) = \operatorname{dist}(x, Y_{\Sigma,c}(R^*) - Z(D))$$

Let $x^*_{\Sigma,c}$ be the point in $Y_{\Sigma,c}(R^*) - Z(D)$ closest to x, and for $x \notin Y_{\Sigma,c}(R^*) - Z(D)$ let $e_{\Sigma,c}(x)$ be the visual limit of the ray from $x^*_{\Sigma,c}$ through x.

Proposition 5.2. *Let $x \in X$ be a point. Let Σ and Σ' be two apartments containing x, let c and c' be two chambers of Δ reducing x. Assume that $c \subset \partial(\Sigma)$ and $c' \subset \partial(\Sigma')$. Then $h_{\Sigma,c}(x) = h_{\Sigma',c'}(x)$. Moreover, $e_{\Sigma,c}(x) = e_{\Sigma',c'}(x) \in \sigma(x)$ provided $h_{\Sigma,c}(x) > 0$.*

Proof. Assume first $\beta_c(x) \leq R^*$. Since $R^* \geq R$, Corollary 1.8 implies $\beta_{c'}(x) \leq R^*$. Hence

$$h_{\Sigma,c}(x) = 0 = h_{\Sigma',c'}(x)\,.$$

It remains to argue the case $\beta_c(x) > R^* \geq R$. First, we work inside Σ. Choose $y \in x + Z(D)$ so that it minimizes the distance to $Y_{\Sigma,c}(R^*)$, and let $y_{\Sigma,c}$ the point in $Y_{\Sigma,c}(R^*)$ closest to y. By choice of R^*, Observation 5.1 applies whence $\sigma_{R^*}(y, c) \subseteq \sigma_R(x, c) = \sigma(x)$. It follows from Observation 1.2 that $y_{\Sigma,c}$ is also the point in $Y_{\Sigma,\sigma(x)}(R^*)$ closest to y.

Now the isometry argument applies: since $\beta_c(x) \geq R^* \geq R$ we have $\sigma(x) \subseteq c \cap c'$. Hence there is a Coxeter isomorphism $\iota : \Sigma \to \Sigma'$ fixing x and $\sigma(x)$. Since $Y_{\Sigma,\sigma(x)}(R^*)$ is only defined in terms of Busemann functions indexed by vertices in $\sigma(x)$, the isometry ι identifies $Y_{\Sigma,\sigma(x)}(R^*)$ with $Y_{\Sigma',\sigma(x)}(R^*)$. As ι is a Coxeter isomorphism, it identifies the two sets $x + Z(D)$ as drawn in Σ and Σ'. It follows that

$$\begin{aligned} h_{\Sigma,c}(x) &= \operatorname{dist}_{\Sigma}(x + Z(D), Y_{\Sigma,\sigma(x)}(R^*)) \\ &= \operatorname{dist}_{\Sigma'}(x + Z(D), Y_{\Sigma',\sigma(x)}(R^*)) \\ &= h_{\Sigma',c'}(x)\,. \end{aligned}$$

If $h_{\Sigma,c}(x) > 0$, then $y_{\Sigma,c} \neq y$ and the ray from $y_{\Sigma,c}$ through y is parallel to the ray from $x^*_{\Sigma,c}$ through x. Hence, it defines the same visual end, which lies in $\sigma_{R^*}(y, c) \subseteq \sigma_R(x, c) = \sigma(x)$. The isometry ι identifies the ray from $y_{\Sigma,c}$ through y with its counter part in Σ'. Hence, $e_{\Sigma,c}(x) = e_{\Sigma',c'}(x)$. □

Observation 5.3. *Note that the ray from $y_{\Sigma,c}$ through y gives the direction of fastest ascent for the function $h_{\Sigma,c}$ in the point x. Also, moving x in that direction increases $h_{\Sigma,c}(x)$ with unit speed and the gradient of $h_{\Sigma,c}$ does not change along this ray.* □

Hence, we can define $h(x) := h_{\Sigma,c}(x)$ and $e(x) := e_{\Sigma,c}(x)$. Here Σ is any apartment of X containing x and c is any chamber in Δ lying in $\partial(\Sigma)$ and reducing x.

Observation 5.4. *Since the reduction datum used in the construction is Γ-invariant, so is the function h.* □

Observation 5.5. *There is a constant C, depending on D and R^*, such that $\hat{h}(x) \leq h(x) + C$ for each $x \in X$. Hence Proposition 2.4 implies that each sublevel set $h^{-1}([0,t]) \subseteq \hat{h}^{-1}([0,t+C])$ has compact quotient modulo the action of Γ.* □

As for continuity and the gradient field, nothing essential changes.

Observation 5.6. *The same reasoning as in the proof of Proposition 2.5 shows that the function h is continuous.* □

Proposition 5.7. *Let x and y be two points in X that lie in a common closed chamber of X. Then*

$$h(y) - h(x) \leq \operatorname{dist}(y, x)$$

with equality if and only if y lies on the ray $[x, e(x))$. In case of equality, moreover $e(y) = e(x)$.

Proof. By uniformity of the reduction datum, choose c and Σ so that h and $h_{\Sigma,c}$ agree on the segment $[x, y]$. Then

$$\begin{aligned} h(y) - h(x) &= h_{\Sigma,c}(y) - h_{\Sigma,c}(x) \\ &= \operatorname{dist}(y, Y_{\Sigma,c}(R^*) - Z(D)) - \operatorname{dist}(x, Y_{\Sigma,c}(R^*) - Z(D)) \\ &\leq \operatorname{dist}(y, x). \end{aligned}$$

By Observation 5.3, we have equality if $y \in [x, e(x))$, and in this case $e(x) = e(y)$. □

For $x \in X$, we define the gradient $\nabla_x h \in \operatorname{lk}(x)$ to be the direction defined by the geodesic ray $[x, e(x))$. Along this ray, the function h increases with unit speed and all other directions show a slower increase. Thus, the geodesic ray $[x, e(x))$ can be regarded as the flow line of the gradient field ∇h starting at x.

Let us call a brick any subset of X that arises as the convex hull of a set of vertices of a common chamber in X.

Observation 5.8. *Let $x \in X$ be a point in a brick B such that $\nabla_x h$ is perpendicular to B. Then x is a point of lowest height in B.*

Proof. Choose Σ and c so that h agrees with $h_{\Sigma,c}$ on B. Hence, h is a convex function on B and the claim follows. □

To actually ensure that h is superior to $\hat{h}$, we have to strengthen the requirement on D. Of course, we have to adjust R^* accordingly.

We call D almost rich if for any two vertices $\mathbf{v}$ and $\mathbf{v}'$ of $\hat{\Sigma}$ that belong to a common chamber, the difference $\mathbf{v} - \mathbf{v}' \in D$. Note that one can obtain a finite, admissible, almost rich set D by starting with the finite set of difference vectors arising from the vertices of a fixed chamber (note that this is automatically symmetric with respect to the origin) and then closing the set with respect to the action of W: since $\mathbf{0}$ is a special vertex, W acts transitively on parallelism classes of chambers in $\hat{\Sigma}$.

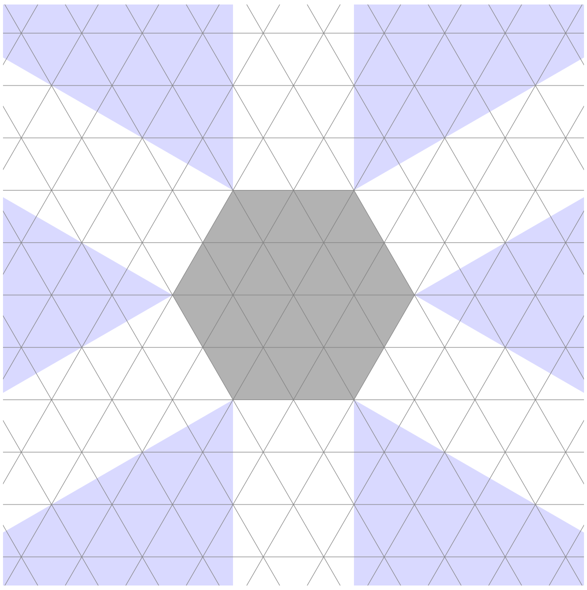

Figure 9: an almost rich zonotope for $\tilde{\mathsf{A}}_2$

The lightly shaded areas and the white corridors are the normal cones for the zonotope.

When D is almost rich, the results of Section 3 apply to bricks. Let us spell out the consequences of Proposition 3.5 combined with Example 3.6.

Corollary 5.9 (to Proposition 3.5). *If D is almost rich, the function h assumes its minimum on any brick in X in a vertex, i.e., the subset of point of minimum height contains a vertex. The subset of points of maximum height, on the other hand, is a face of the brick.* □

Proposition 5.10 (Gradient Criterion). *We still assume that D is almost rich. Let x and y be two distinct vertices in X that lie in a common chamber. Then the following hold:*

1. *The function h is monotonic on the line segment $\varepsilon := [x, y]$.*
2. *The angle $\angle_x(\varepsilon, \nabla_x h) > \frac{\pi}{2}$ if and only if $h(y) < h(x)$.*

Proof. By Corollary 5.9, h attains its minimum along ε at a boundary point. As h is a convex function, this proves the first claim.

If $\angle_x(\varepsilon, \nabla_x h) \neq \frac{\pi}{2}$ the height h changes when one moves from the vertex x infinitesimally into the segment ε. If the angle is obtuse, the height decreases; if the angle is acute, the height increases. By monotonicity, x must be the highest or lowest point on ε, respectively. Observation 5.8 covers the remaining case that $\nabla_x h$ is orthogonal to ε. □

6 Cells of constant height

Let $x \in X$. We think of the link $\mathrm{lk}(x)$ as the space of directions issuing from x. It is a spherical building and we regard it as a metric space via the angular metric.

Now suppose that x is carried by the cell τ. The link $\mathrm{lk}(\tau)$ is also a spherical building. Its simplicial structure corresponds to the poset of cofaces of τ in X. We can realize $\mathrm{lk}(\tau)$ as the space of directions at x orthogonal to τ. For two different points carried by τ, the corresponding realizations of $\mathrm{lk}(\tau)$ are canonically identified and we may think of elements in $\mathrm{lk}(\tau)$ as parallel fields of directions perpendicular to τ. This way, $\mathrm{lk}(\tau)$ carries an angular metric. We thus consider $\mathrm{lk}(\tau)$ as a metric space.

The point link splits as a spherical join

$$\mathrm{lk}(x) = \partial(\tau) * \mathrm{lk}(\tau) \tag{2}$$

where $\partial(\tau)$ is the subspace of $\mathrm{lk}(x)$ consisting of those directions that do not leave τ. It is a round sphere in the angular metric. As τ is a poly-simplicial cell, there is also an obvious cell structure on $\partial(\tau)$.

A cell τ in X is <u>h-flat</u> if h restricts to a constant function on τ.

Observation 6.1. *Let τ be an h-flat cell. Then all flow lines issuing in τ are pairwise parallel and orthogonal to τ.*

Proof. For flow lines issuing from points carried by τ, the claim is clear; and it follows for points on the boundary by continuity. □

Hence, we can talk about the <u>gradient</u> $\nabla_\tau h$ of a h-flat cell as a point in the link $\mathrm{lk}(\tau)$. Regarding the gradient as the north pole in the spherical building $\mathrm{lk}(\tau)$, the link decomposes as in (1)

$$\mathrm{lk}(\tau) = \mathrm{lk}_{\mathrm{hor}}(\tau) * \mathrm{lk}_{\mathrm{ver}}(\tau) \tag{3}$$

into the horizontal and vertical parts of $\mathrm{lk}(\tau)$ relative to the north pole $\nabla_\tau h$. We call the horizontal part $\mathrm{lk}_{\mathrm{hor}}(\tau)$ the <u>horizontal link</u> of the h-flat cell τ, and we call the vertical part $\mathrm{lk}_{\mathrm{ver}}(\tau)$ its <u>vertical link</u>. Beware that the vertical link can contain equatorial cells; and consequently not every h-flat coface of a h-flat cell τ defines a simplex in $\mathrm{lk}_{\mathrm{hor}}(\tau)$. It can also happen that a cell in the horizontal link is not h-flat.

7 Depth

Horizontal cells are the main obstacle for the analysis of the cocompact filtration of X by height. We will use the method of [BW08] as extended to non-irreducible buildings in [Witz10] to cope with this difficulty. Here, we mostly follow [Witz10, Section 2].

Let τ be an h-flat cell in X. By Observation 6.1, the flow lines starting in τ are pairwise parallel geodesic rays in X and therefore, they define a point $e(\tau)$ in the spherical building at infinity. Let β be a Busemann function centered at that point. Since the flow lines are orthogonal to τ, the function β is constant on τ, i.e., the simplex τ is $\underline{\beta\text{-flat}}$. The notion of the horizontal and vertical link of τ defined above agree with the notions in [Witz10, Section 2], whence we can use some results therein directly.

The Busemann function β is not constant on any factor X_p. In the Rank Theorem, the group $\mathcal{G}$ is assumed to be absolutely almost simple. Hence, the factors X_p are all irreducible. It follows that β is not constant on any irreducible factor X, i.e., the Busemann function is in <u>general position</u>, see Proposition 12.2.

Lemma 7.1. *For any h-flat cell τ, there is a unique face $\tau^{\min}$ such that for any proper face $\sigma < \tau$, the following equivalence holds*

$$\tau \textit{ defines a simplex in the horizontal link of } \sigma \quad \textit{if and only if} \quad \tau^{\min} \leq \sigma.$$

Proof. Note that τ is β-flat for any Busemann function β centered at $e(\tau)$. Then the statement follows from [Witz10, Lemma 2.7]. □

In the same way, the following lemma is an immediate consequence of [Witz10, Observation 2.11].

Lemma 7.2. *Suppose $\tau^{\min} \leq \sigma \leq \tau$, i.e., τ defines a simplex in the horizontal link of σ. Then $\tau^{\min} = \sigma^{\min}$.* □

For any two β-flat cells τ and σ, we define <u>going up</u> as

$$\sigma \nearrow \tau \quad :\Longleftrightarrow \quad \sigma = \tau^{\min} \neq \tau$$

and <u>going down</u> as

$$\tau \searrow \sigma \quad :\Longleftrightarrow \quad \tau^{\min} \not\leq \sigma < \tau.$$

We define a <u>β-move</u> as either going up or going down.

Observation 7.3. *If there is a move from τ to τ', then either τ is a face of τ' or τ' is a face of τ. In either case, we have $e(\tau) = e(\tau')$.* □

The following is the statement of [Witz10, Proposition 2.9]:

Proposition 7.4. *There is a uniform upper bound, depending only on the building X, on the length of any sequence of β-moves.* □

We define the <u>depth</u> $\mathrm{dp}(\tau)$ of an h-flat cell τ as the maximum length of a sequence of β-moves starting at τ for the corresponding Busemann function β given by the flow lines of the gradient field on τ.

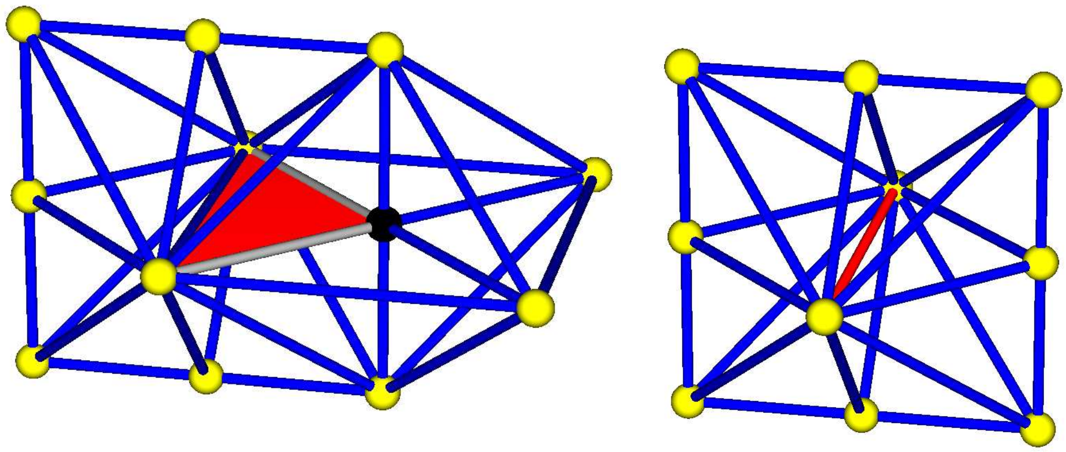

Figure 10: the face $\tau^{\min}$

Both figures take place inside the Coxeter complex $\tilde{\mathsf{B}}_3$. In the picture on the left hand side, the black vertex is the face $\tau^{\min}$ of the horizontal solidly colored 2-simplex τ. The two edges of τ containing $\tau^{\min}$ illustrate Lemma 7.2. In the picture on the right, the horizontal simplex τ is the center edge. Here, we have $\tau = \tau^{\min}$.

8 The Morse function

From now on, we assume that D is almost rich. Let τ be any cell of X. By Proposition 5.9, the max-set of h on τ is a face $\hat{\tau}$, which we call the roof of τ. The roof is h-flat. We define the depth of τ of X as follows:

$$\mathrm{dp}(\tau) := \begin{cases} \mathrm{dp}(\tau) & \text{if } \tau \text{ is flat} \\ \mathrm{dp}(\hat{\tau}) - \frac{1}{2} & \text{otherwise} \end{cases}$$

We define the following Morse function on cells of X:

$$\begin{array}{rcl} f : \mathcal{C}(X) & \longrightarrow & \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ \tau & \mapsto & \left(\max\limits_{\tau} (h) \, , \mathrm{dp}(\tau) \, , \dim(\tau) \right) \end{array}$$

Observation 8.1. *The dimension component assures that comparable but distinct cells (i.e., one is a strict face of the other or vice versa) are not assigned the same triple.* $\square$

The cells of X are in one-to-one correspondence to the vertices of the barycentric subdivision $\mathring{X}$ of X. Ordering $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ lexicographically, we regard f as a Morse function on $\mathring{X}$.

For each cell τ let $\mathring{\tau}$ denote its barycenter, i.e., the vertex of $\mathring{X}$ that corresponds to τ. The link of the vertex $\mathring{\tau}$ in $\mathring{X}$ decomposes as a join of the boundary $\partial(\tau)$ and the link $\mathrm{lk}(\tau)$. This corresponds to the decomposition (2).

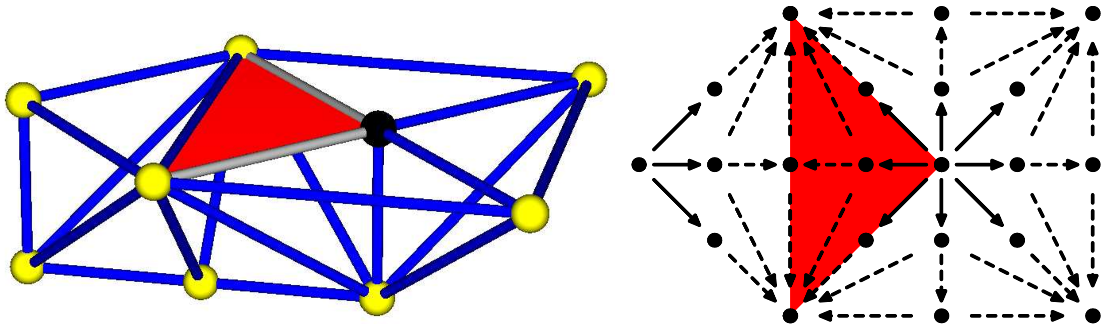

Figure 11: sequences of moves

This figure continues Figure 10; it also takes place inside a Coxeter complex of type $\tilde{\mathsf{B}}_3$. It shows a possible patch of horizontal 2-cells. Each dot in the picture on the right represents a cell; for orientation, one horizontal 2-cells has been filled in. Arrows indicate moves: solid arrows represent going up, whereas dashed arrows represent going down. Note that there are no moves between triangles and short edges.

The descending link of a vertex $\mathring{\tau} \in \mathring{X}$ consists of those simplices in the link all of whose vertices lie strictly below $\mathring{\tau}$ with respect to the Morse function f. Hence, the descending link also decomposes as a join:

$$\mathrm{lk}^{\downarrow}(\mathring{\tau}) = \partial^{\downarrow}(\tau) * \mathrm{lk}^{\downarrow}(\tau) \tag{4}$$

Here, $\partial^{\downarrow}(\tau)$ can be regarded as the poset of strict faces of τ with smaller f-value and $\mathrm{lk}^{\downarrow}(\tau)$ can likewise be viewed as the poset of strict cofaces with smaller f-value.

9 Descending links

We call τ insignificant if $\tau \neq \hat{\tau}^{\min}$ and significant otherwise. We shall deal with the insignificant cells first. Here, the descending link is always contractible. In fact, in the decomposition $\mathrm{lk}^{\downarrow}(\mathring{\tau}) = \partial^{\downarrow}(\tau) * \mathrm{lk}^{\downarrow}(\tau)$, already the boundary part $\partial^{\downarrow}(\tau)$ is contractible.

Proposition 9.1. *If $\tau \neq \hat{\tau}^{\min}$ then $\partial^{\downarrow}(\tau)$ is contractible. More precisely, the complex $\partial^{\downarrow}(\tau)$ deformation retracts onto the subcomplex $\partial(\tau) \setminus \mathrm{st}\left(\hat{\tau}^{\min}\right)$.*

Proof. First, we note that $\hat{\tau}^{\min}$ cannot correspond to a vertex in the descending link. The height does not decide as

$$\max_{\tau}(h) = \max_{\hat{\tau}}(h) = \max_{\hat{\tau}^{\min}}(h)\,.$$

As for the depth, we have

$$\mathrm{dp}(\tau) \leq \mathrm{dp}(\hat{\tau}) \leq \mathrm{dp}\left(\hat{\tau}^{\min}\right).$$

If $\tau \neq \hat{\tau}$, then the first inequality is strict. If $\tau = \hat{\tau}$, the hypothesis that τ is insignificant implies $\hat{\tau} \neq \hat{\tau}^{\min}$ whence there is a move $\hat{\tau}^{\min} \nearrow \hat{\tau}$ and $\mathrm{dp}(\hat{\tau}) < \mathrm{dp}\left(\hat{\tau}^{\min}\right)$.

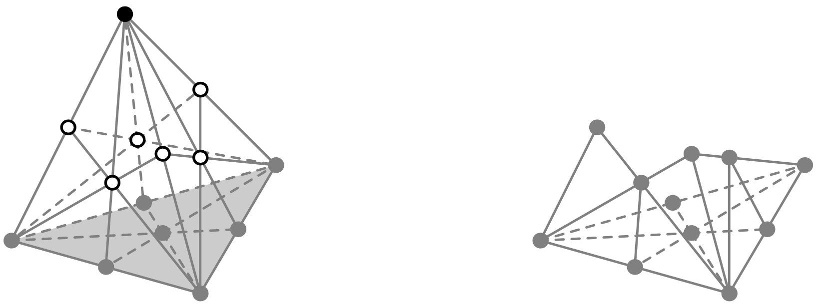

Figure 12: the deformation of $\partial^{\downarrow}(\tau)$ onto $\partial(\tau) \setminus \mathrm{st}\big(\hat{\tau}^{\min}\big)$ from Proposition 9.1
On the left hand, the solid vertex at the top is $\hat{\tau}^{\min}$. The shaded triangle is $\partial(\tau)\setminus\mathrm{st}\big(\hat{\tau}^{\min}\big)$ and guaranteed to be descending. Note that the barycentric subdivision is drawn but $\mathrm{st}\big(\hat{\tau}^{\min}\big)$ denotes the open star of $\hat{\tau}^{\min}$ with respect to the original cell structure. The hollow vertices are *unknown*; they could be descending or ascending. On the right hand, one possibility for $\partial^{\downarrow}(\tau)$ is shown.

In either case the strict inequality $\mathrm{dp}(\tau) < \mathrm{dp}\big(\hat{\tau}^{\min}\big)$ follows, whence $f(\tau) < f\big(\hat{\tau}^{\min}\big)$, i.e., $\hat{\tau}^{\min}$ is not descending.

We now turn to the "opposite part", i.e., we identify a subcomplex of $\partial(\tau)$ that is descending. Let σ be a cell with $\hat{\tau}^{\min} \not\leq \sigma < \tau$. As $\sigma < \tau$ we have $\max_{\sigma}(h) \leq \max_{\tau}(h)$. Also, $\hat{\tau}^{\min} \not\leq \sigma$ implies $\hat{\tau}^{\min} \not\leq \hat{\sigma}$. Hence, there is a move $\hat{\tau} \searrow \hat{\sigma}$ whence $\mathrm{dp}(\hat{\sigma}) < \mathrm{dp}(\hat{\tau})$. Since depths of flat cells are integer valued, $\mathrm{dp}(\sigma) \leq \mathrm{dp}(\hat{\sigma}) < \mathrm{dp}(\hat{\tau}) - \frac{1}{2} \leq \mathrm{dp}(\tau)$.

So, let K be the subcomplex of $\partial(\tau)$ spanned by vertices $\mathring{\sigma}$ with $\hat{\tau}^{\min} \not\leq \sigma < \tau$. We have seen that K is descending. Let v be the barycenter of $\hat{\tau}^{\min}$. We have seen that v is ascending, not descending. Radial projection inside τ away from v defines a deformation retraction of $\partial^{\downarrow}(\tau)$ onto K. Since K is a sphere with an open star of a cell removed, $\partial^{\downarrow}(\tau)$ is contractible. $\square$

Corollary 9.2. *If τ is insignificant then the descending link $\mathrm{lk}^{\downarrow}(\mathring{\tau})$ of its barycenter is contractible.* $\square$

The remainder of this section is devoted to the analysis of descending links $\mathrm{lk}^{\downarrow}(\mathring{\tau})$ when τ is a significant cell.

Observation 9.3. *If τ is significant, i.e., $\tau = \hat{\tau}^{\min}$, then τ is flat whence $\max_{\sigma}(h) = \max_{\tau}(h)$ for any face $\sigma \leq \tau$. In particular, the depth and the dimension determine which part of $\partial(\tau)$ is descending.*

It follows that $\partial(\tau)$ is completely descending: for any proper face $\sigma < \tau$, there is a move $\tau \searrow \sigma$ whence $\mathrm{dp}(\sigma) < \mathrm{dp}(\tau)$. Thus, $\partial^{\downarrow}(\tau)$ is a sphere of dimension $\dim(\tau) - 1$. $\square$

Observation 9.4. *If $\xi > \tau = \hat{\tau}^{\min}$ is a flat coface of a significant cell, then either $\tau \nearrow \xi$ or $\xi \searrow \tau$: If $\xi^{\min} \leq \tau$, then $\xi^{\min} = \tau^{\min} = \tau$ by Lemma 7.2. In this case, $\tau \nearrow \xi$. If $\xi^{\min} \not\leq \tau$, then $\xi \searrow \tau$.* $\square$

Proposition 9.5. *Assume that τ is significant. Fix cofaces ξ and ζ with $\tau < \xi \leq \zeta$. If $f(\zeta) < f(\tau)$, then $f(\xi) < f(\tau)$.*

Proof. First note that $\max_\tau (h) \leq \max_\xi (h) \leq \max_\zeta (h)$ as $\tau < \xi \leq \zeta$. By hypothesis, $\max_\zeta (h) \leq \max_\tau (h)$. Thus, we have equality throughout.

As $\dim(\zeta) > \dim(\tau)$, the hypothesis $f(\zeta) < f(\tau)$ implies $\mathrm{dp}(\zeta) < f(\tau)$. Passing to roofs, we have the inclusions $\tau \leq \hat{\xi} \leq \hat{\zeta}$ of flat cells. If $\tau = \hat{\xi}$ then $\xi \neq \hat{\xi}$. Hence, $\mathrm{dp}(\xi) < \mathrm{dp}\left(\hat{\xi}\right) = \mathrm{dp}(\tau)$ and ξ is descending.

If, on the other hand, $\tau \neq \hat{\xi}$ then $\hat{\zeta}$ is a proper flat coface of the significant cell τ. By Observation 9.4, there is a move $\tau \nearrow \hat{\zeta}$ or a move $\hat{\zeta} \searrow \tau$. In the latter case, $\mathrm{dp}(\zeta) \geq \mathrm{dp}\left(\hat{\zeta}\right) > \mathrm{dp}(\tau)$ contradicting the hypothesis that $f(\zeta) < f(\tau)$. Therefore, there is a move $\tau \nearrow \hat{\zeta}$, that is, $\tau = \hat{\zeta}^{\min}$. Then Lemma 7.2 implies $\tau = \hat{\xi}^{\min}$ whence $\tau \nearrow \hat{\xi}^{\min}$ and $\mathrm{dp}(\xi) \leq \mathrm{dp}\left(\hat{\xi}\right) < \mathrm{dp}(\tau)$. □

Proposition 9.5 justifies a notational vagueness of which we are guilty. In Section 6, particularly in the decomposition 2, we used $\mathrm{lk}(\tau)$ to denote a spherical building. In Section 8, we switched to its barycentric subdivision (the geometric realization of the poset of strict cofaces of τ). Since barycenters of insignificant cells have contractible descending links just by their boundary part $\partial^{\downarrow}(\tau)$, the precise structure of $\mathrm{lk}^{\downarrow}(\tau)$ did not matter in this case. If τ is significant, $\mathrm{lk}^{\downarrow}(\tau)$ does matter. Although it is defined as a subcomplex of the barycentric subdivision, Proposition 9.5 implies that we can regard the descending link as a subcomplex of the spherical building $\mathrm{lk}(\tau)$. Hence, we put:

$$\begin{aligned} \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau) &:= \mathrm{lk}^{\downarrow}(\tau) \cap \mathrm{lk}_{\mathrm{hor}}(\tau) \\ \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau) &:= \mathrm{lk}^{\downarrow}(\tau) \cap \mathrm{lk}_{\mathrm{ver}}(\tau) \end{aligned}$$

Proposition 9.6. *If τ is significant, $\mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$ is an open hemisphere complex in $\mathrm{lk}(\tau)$ with respect to the north pole $\nabla_\tau h$.*

Proof. Let H be the open hemisphere complex of $\mathrm{lk}(\tau)$ with respect to $\nabla_\tau h$. The Gradient Criterion 5.10 implies $H \subseteq \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$. To show equality, it suffices to argue that no flat coface $\xi > \tau$ in $\mathrm{lk}_{\mathrm{ver}}(\tau)$ is descending, i.e., $f(\xi) > f(\tau)$. However, as ξ belongs to $\mathrm{lk}_{\mathrm{ver}}(\tau)$, it does not belong to $\mathrm{lk}_{\mathrm{hor}}(\tau)$ whence, by Lemma 7.1, there is a move $\xi \searrow \tau$. Thus, $\mathrm{dp}(\xi) > \mathrm{dp}(\tau)$. As ξ is flat, $f(\xi) > f(\tau)$ follows. □

Proposition 9.7. *The decomposition 1 at the end of Section 4 induces the decomposition*

$$\mathrm{lk}^{\downarrow}(\tau) = \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau) * \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$$

provided τ is significant.

Proof. As each vertex of $\mathrm{lk}^{\downarrow}(\tau)$ lies in $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$ or $\mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$, it follows that $\mathrm{lk}^{\downarrow}(\tau) \subseteq \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau) * \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$.

To see the converse, let ξ_{h} and ξ_{v} denote strict cofaces of τ where ξ_{h} determines a simplex in $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$ and ξ_{v} determines a simplex in $\mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$. We need to show that the join $\xi := \xi_{\mathrm{h}} * \xi_{\mathrm{v}}$ lies in $\mathrm{lk}^{\downarrow}(\tau)$, i.e., $f(\xi) < f(\tau)$. The cell ξ is the smallest coface of τ containing ξ_{h} and ξ_{v}.

Since ξ_{h} is descending, it is flat. As $\mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$ is an open hemisphere complex, all directions from τ that have a non-vanishing component into a direction of ξ_{v} are descending with respect to the height h. Hence $\xi_{\mathrm{h}} = \hat{\xi}$ and $\max_{\xi}(h) = \max_{\xi_{\mathrm{h}}}(h) = \max_{\tau}(h)$. Since ξ_{h} is descending, $\mathrm{dp}(\xi) \leq \mathrm{dp}(\xi_{\mathrm{h}}) < \mathrm{dp}(\tau)$. Thus, $f(\xi) < f(\tau)$. □

Corollary 9.8. *For significant τ, the descending link $\mathrm{lk}^{\downarrow}(\mathring{\tau})$ decomposes as*

$$\mathrm{lk}^{\downarrow}(\mathring{\tau}) = \partial(\tau) * \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau) * \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)\,. \qquad \square$$

In Proposition 9.6, we have determined that $\mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau)$ is an open hemisphere complex. It remains to analyze $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$.

Lemma 9.9. *Let $\xi \in \mathrm{lk}_{\mathrm{hor}}(\tau)$ for a significant cell τ, i.e., ξ is a proper coface of τ with $\xi^{\min} \leq \tau = \tau^{\min} < \xi$. Then, the following are equivalent:*

1. *The cell ξ is descending, i.e., $\xi \in \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$.*
2. *The cell ξ is h-flat.*
3. *We have $\max_{\xi}(h) = \max_{\tau}(h)$.*

Proof. First assume that ξ is flat. Then $\max_{\xi}(h) = \max_{\tau}(h)$. Also, by Lemma 7.2, $\xi^{\min} = \tau^{\min} = \tau$. Hence, there is a move $\tau \nearrow \xi$ whence $\mathrm{dp}(\xi) < \mathrm{dp}(\tau)$. Hence $f(\xi) < f(\tau)$.

Now assume that ξ is not flat. The Gradient Criterion 5.10 implies that $\max_{\xi}(h) > \max_{\tau}(h)$. In particular, $f(\xi) > f(\tau)$. □

For the last part of the analysis, we need to strengthen the hypothesis on D one last time. We call D <u>rich</u> if it contains the differences $\mathbf{v}-\mathbf{v}'$ of any to vertices $\mathbf{v}, \mathbf{v}' \in \hat{\Sigma}$ whose closed stars intersect.

Also, at last, we have to enlarge the diameter d of uniformity for the reduction datum. Using Theorem 1.9, we assume that any closed star of any cell can be uniformly reduced. This affects the constants r and R. Using a rich D, the construction of Section 5 then will yield an appropriate R^*.

Let τ be a significant cell, let c be a chamber in Δ uniformly reducing the closed star of τ, and let Σ be an apartment containing τ with $c \subset \partial(\Sigma)$. Put:

$$L^{\uparrow}_{\Sigma}(\tau) := \{v \in \Sigma \mid v \text{ is a vertex, } v \vee \tau \text{ defines a cell in } \mathrm{lk}(\tau)\,,\ h(v) > h(\tau)\}$$

Let A_{Σ} denote the convex hull of $L^{\uparrow}_{\Sigma}(\tau)$ in the euclidean space Σ.

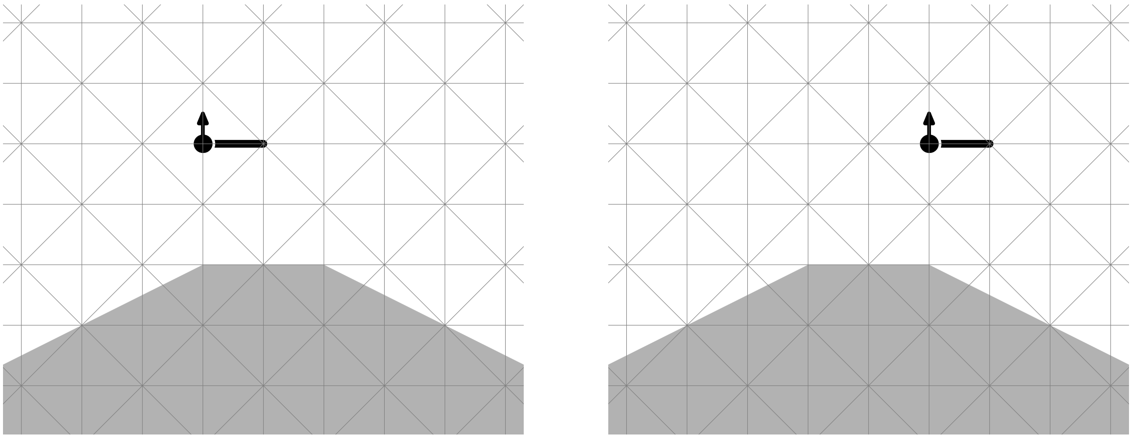

Figure 13: a flat vs. a non-flat horizontal coface

The primary height is distance from the shaded area, the significant cell is the fat vertex, the little arrow indicates the gradient of the height function, and the horizontal coface is the marked edge issuing to the right. To see that the edge is in the horizontal link of its left vertex, recall that the link decomposes as a join into the vertical and horizontal parts. The horizontal part is the maximal join factor that is perpendicular to the gradient.

Observation 9.10. *Assume that D is rich. Then A_Σ is a convex polytope satisfying the hypotheses of Proposition 3.5. Hence, h assumes its minimum on A_Σ in a vertex, which is still higher than the flat cell τ.* □

Corollary 9.11. *Provided that D is rich, A_Σ is disjoint from the affine subspace of Σ spanned by τ.* □

The convex set A_Σ induces a closed subset $\tilde{A}_\Sigma$ in $\mathrm{lk}_\Sigma(\tau) \subset \mathrm{lk}(\tau)$ by projection onto an orthogonal complement of the span of τ.

Corollary 9.12. *Also under the hypothesis that D is rich, the subset $\tilde{A}_\Sigma \subset \mathrm{lk}_\Sigma(\tau)$ is closed, convex, and has diameter strictly less than π.* □

We can extract a little more information:

Observation 9.13. *By Lemma 9.9, a horizontal coface ξ of τ is descending if and only if it is flat. Hence, the value of h on ξ cannot exceed the value on τ. Therefore, if D is rich, the descending horizontal link $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$ and $\tilde{A}_\Sigma$ are disjoint by Observation 9.10.* □

Let Σ' be another apartment in X containing τ and satisfying $c \subset \partial(\Sigma')$.

Observation 9.14. *Any Coxeter isomorphism $\iota : \Sigma \to \Sigma'$ that is the identity on the intersection $\Sigma \cap \Sigma'$ makes the diagram*

$$\begin{tikzcd} \Sigma \arrow[rr, "\iota"] \arrow[dr, "h_{\Sigma,c}"'] & & \Sigma' \arrow[dl, "h_{\Sigma',c}"] \\ & \mathbb{R} & \end{tikzcd}$$

commute.

Proof. The height h is defined in terms of (a) Busemann functions β_v for points $v \in c$ and (b) zonotopes of the form $\mathbf{x} + Z(D)$. The Busemann functions are clearly preserved under the Coxeter isomorphism ι since the intersection $\Sigma \cap \Sigma'$ contains a sector bounding c. Since ι is a Coxeter isomorphism, it also preserves D which is invariant under the full spherical Weyl group. $\square$

Now, we fix a chamber C in the visual boundary $\partial(X)$ that contains c. If Σ and Σ' are two apartments both containing the convex cone $\overline{\tau, C}$ then the retraction $\rho : X \to \Sigma$ of the building X onto Σ centered at the chamber C restricts to a Coxeter isomorphism $\rho|_{\Sigma'} : \Sigma' \to \Sigma$ to which Observation 9.14 applies. Now, we use the hypothesis that c uniformly reduces the closed star of τ. Hence:

$$\begin{aligned} h|_{\mathrm{st}(\tau)\cap\Sigma} &= h_{\Sigma,c}|_{\mathrm{st}(\tau)\cap\Sigma} \\ h|_{\mathrm{st}(\tau)\cap\Sigma'} &= h_{\Sigma',c}|_{\mathrm{st}(\tau)\cap\Sigma'} \end{aligned}$$

In particular, $\rho|_{\Sigma'}$ identifies $A_{\Sigma'}$ with A_Σ.

Observation 9.15. *We assume that D is rich so that we can use the previous results. Let $\tilde{C}$ be the projection of C in the spherical building $\mathrm{lk}_{\mathrm{hor}}(\tau)$. It is a chamber. Let $\tilde{\Sigma}$ be the apartment $\Sigma \cap \mathrm{lk}_{\mathrm{hor}}(\tau)$. Then $\tilde{C} \subset \tilde{\Sigma}$. Let $\tilde{\rho} : \mathrm{lk}_{\mathrm{hor}}(\tau) \to \tilde{\Sigma}$ be the retraction onto $\tilde{\Sigma}$ centered at $\tilde{C}$. Put $\tilde{A} := \tilde{\rho}^{-1}\left(\tilde{A}_\Sigma\right)$. Then, any apartment in $\mathrm{lk}_{\mathrm{hor}}(\tau)$ that contains $\tilde{C}$ is of the form $\tilde{\Sigma}' := \Sigma' \cap \mathrm{lk}_{\mathrm{hor}}(\tau)$ where Σ' is an apartment in X containing the convex cone $\overline{\tau, C}$; moreover $\tilde{A} \cap \Sigma' = \tilde{A}_{\Sigma'}$ is a closed convex subset of $\tilde{\Sigma}'$ of diameter less than π by Corollary 9.12.*

Hence Corollary 4.5 applies; and the the maximal subcomplex of the complement $\mathrm{lk}_{\mathrm{hor}}(\tau) \setminus \tilde{A}$ is $(\dim(\mathrm{lk}_{\mathrm{hor}}(\tau)) - 1)$-connected and of dimension $\dim(\mathrm{lk}_{\mathrm{hor}}(\tau))$. $\square$

Corollary 9.16. *Assume that D is rich. The horizontal descending link $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$ of a significant cell τ is contractible or spherical of dimension $\dim(\mathrm{lk}_{\mathrm{hor}}(\tau))$.*

Proof. By the preceding Observation 9.15, we have to argue that $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$ is the maximal subcomplex of $\mathrm{lk}_{\mathrm{hor}}(\tau) \setminus \tilde{A}$. Observation 9.13 implies the inclusion $\mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau) \subseteq \mathrm{lk}_{\mathrm{hor}}(\tau) \setminus \tilde{A}$. On the other hand, any vertex of $\mathrm{lk}_{\mathrm{hor}}(\tau) \setminus \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$ lies within $\tilde{A}$ by definition of the sets $L^{\uparrow}_{\Sigma}(\tau)$. $\square$

We can summarize the analysis of descending links:

Proposition 9.17. *Assume that D is rich. Then the descending link $\mathrm{lk}^{\downarrow}(\mathring{\tau})$ of any barycenter is contractible or spherical of dimension $\dim(X) - 1$.*

Proof. If τ is insignificant, then $\mathrm{lk}^{\downarrow}(\tau)$ is contractible by Corollary 9.2.

If τ is significant, then the descending link decomposes as

$$\mathrm{lk}^{\downarrow}(\mathring{\tau}) = \partial(\tau) * \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(\tau) * \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(\tau)$$

by Corollary 9.8. The part $\partial(\tau)$ is a sphere of dimension $\dim(\tau)-1$ (or empty if τ is a vertex). The other parts are treated in Proposition 9.6 and Corollary 9.16. Their join is contractible or spherical of dimension $\dim(X)-\dim(\tau)-1$. Hence, $\mathrm{lk}^{\downarrow}(\mathring{\tau})$ is contractible or spherical of dimension $\dim(X)-1$. $\square$

Observation 9.18. *A vertex x is always a significant cell as $x = x^{\min}$ since $x^{\min}$ is a non-empty face of x. Also, a vertex has empty boundary. Hence*

$$\mathrm{lk}^{\downarrow}(\mathring{x}) = \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(x) * \mathrm{lk}^{\downarrow}_{\mathrm{hor}}(x)\,.$$

Generically, $\mathrm{lk}(x)$ will not have a horizontal component: the gradient $\nabla_x h$ will be in general position. In those cases, $\mathrm{lk}^{\downarrow}(\mathring{x}) = \mathrm{lk}^{\downarrow}_{\mathrm{ver}}(x)$ is an (open) hemisphere complex, which is not *contractible by Proposition 4.6 since the building X is thick (in the Rank Theorem, the group $\mathcal{G}$ is assumed to be non-commutative). In particular, there exist arbitrary high vertices with non-contractible descending links.* $\square$

10 Proof of the Rank Theorem

We assume that D is rich and invariant under the full spherical Weyl group. E.g., one could choose D to consist of difference vectors of any pair of vertices in $\hat{\Sigma} = \mathbb{E}$ whose closed stars intersect.

Observation 10.1. *The S-arithmetic group Γ acts on the product X by cell-permuting homeomorphisms. Cell stabilizers are finite.* $\square$

Observation 10.2. *The function f is Γ-invariant by Observation 5.4, and its sub-level complexes are Γ-cocompact by Observation 5.5.* $\square$

Proof of the Rank Theorem. Given the topological properties of descending links, the deduction of finiteness properties is routine.

Since Γ acts cocompactly, there are only finitely many Γ-orbits of cells in X below any given f-bound in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. In particular, only finitely many elements in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ arise as values of f below any given bound. Define $F(i)$ to be the subcomplex of $\mathring{X}$ spanned by the barycenters $\mathring{\tau}$ of cells τ for which there are at most i values in the image $\mathrm{im}(f)$ that are strictly below $f(\tau)$.

By Observation 8.1, there are no f-flat edges in $\mathring{X}$. Thus, $F(i+1)\setminus F(i)$ does not contain adjacent vertices. For any vertex $\mathring{\tau} \in F(i+1)\setminus F(i)$, the descending link $\mathrm{lk}^{\downarrow}(\mathring{\tau})$ is precisely the relative link $\mathrm{lk}(\mathring{\tau}) \cap F(i)$. This relative link is contractible or spherical of dimension $\dim(X)-1$ by Proposition 9.17. Thus, the complex $F(i+1)$ is obtained from $F(i)$ up to homotopy equivalence by attaching d-cells – recall that d is the dimension of X. Observation 9.18 ensures that the extension is nontrivial at infinitely many stages.

The group Γ acts on $\mathring{X}$ by cell-permuting homeomorphisms and with finite cell stabilizers. Thus, all hypotheses of Brown's criterion [Bro87, Corollary 3.3] are satisfied and Γ is of type F_{d-1} but not of type F_d. $\square$

11 Reduction theory: the adelic version

In this section, we describe Harder's version of reduction theory for reductive groups over global function fields. Thus, we relax the hypotheses of the Rank Theorem: the group scheme $\mathcal{G}$ is assumed to be connected and reductive. After Theorem 11.2, we shall add the requirement that $\mathcal{G}$ be K-isotropic.

Let k be the finite field of constants of the global function field K. For any place p on K, let K_p be the completion of K at p. The field K_p is a local field on which we can regard p as a normalized discrete valuation. Let $\mathcal{O}_p$ be the corresponding valuation ring and $\mathfrak{m}_p$ its unique maximal ideal. The residue field $k_p := \mathcal{O}_p/\mathfrak{m}_p$ is a finite extension of the field k of constants. Let $d_p := [k_p : k]$ denote its degree. The modulus map

$$\begin{array}{rcl} \| - \|_p : K_p & \longrightarrow & \mathbb{R} \\ f & \mapsto & |k|^{-d_p p(f)} \end{array}$$

describes how multiplication by f changes the Haar measure on K_p.

For any finite set of places S, the product

$$\mathbb{A}_S := \prod_{p \in S} K_p \quad \times \quad \prod_{p \notin S} \mathcal{O}_p$$

is the ring of S-adeles. Note that the functor $S \mapsto \mathbb{A}_S$ is a directed system indexed by the family of finite sets of places. The ring $\mathbb{A}$ of adeles is by definition the direct limit of this system. As each $\mathbb{A}_S$ is a topological ring, so is $\mathbb{A}$, and $\mathcal{O} := \mathbb{A}_\emptyset = \prod_p \mathcal{O}_p$ is a compact subring.

For any adele $\mathbf{f} = (f_p)_p \in \mathbb{A}$ we define the idele norm as

$$\|\mathbf{f}\| := \prod_p \|f_p\|_p.$$

Taking logarithms, we obtain:

$$\log_{|k|}(\|\mathbf{f}\|) = \sum_p -d_p p(f_p) \tag{5}$$

For any $f \in K$, there are only finitely many places p for which $f \notin \mathcal{O}_p$. Hence, K diagonally embeds into $\mathbb{A}$, and with respect to this inclusion, $\mathcal{O}_S = \mathbb{A}_S \cap K$. Also, the idele norm is trivial on K^*, i.e., we have the product formula

$$\|f\| = \prod_p \|f\|_p = 1 \qquad \text{for any } f \in K^*. \tag{6}$$

Let K' be a finite Galois extension of K. In particular, K' is a global function field in its own right. Let $\mathbb{A}'$ denote the ring of adeles associated to K'. Since every discrete valuation on K extends to at least one valuation on K', there is a diagonal embedding $\mathbb{A} \subseteq \mathbb{A}'$. Let $N : K' \to K$ denote the norm map. For any $f' \in K'$ one has

$$\|N(f')\|_p = \prod_{p' \text{ extends } p} \|f'\|_{p'}. \tag{7}$$

As the idele norm is defined in terms of the modulus maps, we infer:

$$\|N(f')\|_K = \|f'\|_{K'}.$$

Let $\mathcal{G}$ be a reductive group defined over K. Then $\mathcal{G}(\mathbb{A}_S) = \prod_{p\in S}\mathcal{G}(K_p) \times \prod_{p\in S}\mathcal{G}(\mathcal{O}_p)$ and $\mathcal{G}(\mathbb{A}) = \varinjlim_S \mathcal{G}(\mathbb{A}_S)$. Let $\mathfrak{Mult}$ denote the multiplicative group regarded as a group scheme over K. A character on a K-group is a homomorphism into $\mathfrak{Mult}$. One defines:

$$\begin{aligned} \mathcal{G}(\mathbb{A})^\circ &:= \{\gamma \in \mathcal{G}(\mathbb{A}) \mid \|\chi(\gamma)\| = 1 \text{ for any character } \chi : \mathcal{G} \to \mathfrak{Mult} \text{ defined over } K\} \\ \mathcal{G}(\mathbb{A}')^\circ &:= \{\gamma \in \mathcal{G}(\mathbb{A}') \mid \|\chi(\gamma)\| = 1 \text{ for any character } \chi : \mathcal{G} \to \mathfrak{Mult} \text{ defined over } K'\} \end{aligned}$$

Of course, there may be more characters defined over K' than there are defined over K. Hence the latter group appears smaller in this regard. However using the norm map N to average over K'-characters on $\mathcal{G}$, one can deduce from (7) that the inclusion $\mathcal{G}(\mathbb{A}) \leq \mathcal{G}(\mathbb{A}')$ induced by $\mathbb{A} \subseteq \mathbb{A}'$ restricts to an inclusion of $\mathcal{G}(\mathbb{A})^\circ$ in $\mathcal{G}(\mathbb{A}')^\circ$ as a closed topological subgroup.

Lemma 11.1. *The inclusion $\mathcal{G}(\mathbb{A}) \subseteq \mathcal{G}(\mathbb{A}')$ induces proper maps $\mathcal{G}(\mathbb{A})/\mathcal{G}(K) \to \mathcal{G}(\mathbb{A}')/\mathcal{G}(K')$ and $\mathcal{G}(\mathbb{A})^\circ/\mathcal{G}(K) \to \mathcal{G}(\mathbb{A}')^\circ/\mathcal{G}(K')$.*

Proof. This follows from [Hard69, Lemma 2.2.3]. □

The following statement says everything there is to say (from the reduction theory point of view) about K-anisotropic groups:

Theorem 11.2 ([Hard69, Korollar 2.2.7]). *$\mathcal{G}$ is K-anisotropic if and only if $\mathcal{G}(\mathbb{A})^\circ/\mathcal{G}(K)$ is compact.* □

From now on, we assume that $\mathcal{G}$ is K-isotropic.

Note that $\mathcal{G}(\mathcal{O}_p)$ is an open compact subgroup of $\mathcal{G}(K_p)$. Following Harder, we call a subgroup $\mathcal{C}$ of $\mathcal{G}(\mathbb{A})$ standard if $\mathcal{C}$ is of the form $\prod_p \mathcal{C}_p$ where each $\mathcal{C}_p$ is an open compact subgroup of $\mathcal{G}(K_p)$. In particular, the canonical subgroup $\mathcal{G}(\mathcal{O}) = \prod_p \mathcal{G}(\mathcal{O}_p)$ is standard. Let $\mathcal{P}$ be a K-parabolic subgroup with unipotent radical $\mathcal{R}_{\mathrm{u}}$. Starting with a non-vanishing volume form ω on $\mathcal{R}_{\mathrm{u}}$ (in the sense of algebraic geometry and defined over K), the associated measure $\mathrm{d}\,\omega_{\mathbb{A}}$ on $\mathcal{R}_{\mathrm{u}}(\mathbb{A})$ is independent of ω because of the product formula [Weil82, Theorem 2.3.1]; in fact, $\mathrm{d}\,\omega_{\mathbb{A}}$ is proportional to the Tamagawa measure. Harder defines for any parabolic $\mathcal{P}$ and any standard subgroup $\mathcal{C}$:

$$\pi(\mathcal{P}, \mathcal{C}) := \mathrm{vol}_{\mathrm{d}\,\omega_{\mathbb{A}}}(\mathcal{R}_{\mathrm{u}}(\mathbb{A}) \cap \mathcal{C}) \tag{8}$$

As the measure $\mathrm{d}\,\omega_{\mathbb{A}}$ is canonical, this definition is invariant under the conjugacy action of $\mathcal{G}(K)$ on $\mathcal{G}(\mathbb{A})$, i.e., for each element $\gamma \in \mathcal{G}(K)$ we have:

$$\pi(\mathcal{P}, \mathcal{C}) = \pi({}^\gamma\mathcal{P}, {}^\gamma\mathcal{C}) \tag{9}$$

The unipotent radical $\mathcal{R}_{\mathrm{u}}$ is a weight space for the adjoint representation of the parabolic group $\mathcal{P}$. We call the associated character $\chi_{\mathcal{P}} : \mathcal{P} \to \mathfrak{Mult}$ the canonical character of $\mathcal{P}$. Its idele norm is the functional determinant of the conjugacy action of $\mathcal{P}$ on $\mathcal{R}_{\mathrm{u}}$. Hence:

Proposition 11.3 (Transformation Formula [Hard69, Satz 1.3.2]). *For any standard subgroup $\mathcal{C} \leq \mathcal{G}(\mathbb{A})$ and any $\gamma \in \mathcal{P}(\mathbb{A})$, we have*

$$\pi(\mathcal{P}, \mathcal{C}) = \pi(\mathcal{P}, {}^{\gamma}\mathcal{C}) \, \|\chi_{\mathcal{P}}(\gamma)\,\| . \tag{10}$$

Construction 11.4 ([Hard69, page 47]). Assume that $\mathcal{P}$ is a minimal K-parabolic, let $\mathcal{R}$ be its radical, $\mathcal{R}_{\mathrm{u}}$ be its unipotent radical, and put $\mathcal{T} := \mathcal{R}/\mathcal{R}_{\mathrm{u}}$. Let $\mathcal{T}' \leq \mathcal{T}$ be the maximal K-split torus. We think of $\mathcal{T}$ and $\mathcal{T}'$ not just as abstract tori but as tori inside of a Levi subgroup $\mathcal{L}$ of $\mathcal{P}$. In particular, $\mathcal{T}$ is a maximal torus in $\mathcal{L}$ and $\mathcal{T}'$ is a maximal K-split torus inside $\mathcal{L}$. Let $\{\alpha_1, \ldots, \alpha_r\} \subset \mathrm{X}(\mathcal{T}')$ be the set of the simple roots on $\mathcal{T}'$. With $\mathrm{X}(\mathcal{P}) := \mathrm{Hom}_K(\mathcal{P}; \mathfrak{Mult})$, we have $\mathrm{X}(\mathcal{P}) \otimes \mathbb{R} = \mathrm{X}(\mathcal{T}') \otimes \mathbb{R}$ hence, we can regard each α_i as an element of $\mathrm{X}(\mathcal{P}) \otimes \mathbb{R}$. The minimal parabolic $\mathcal{P}$ corresponds to a chamber of the spherical building $\Delta = \Delta_K$. The roots α_i correspond to faces. Hence i can be regarded as a cotype. Let $\mathcal{P}_i$ be the maximal parabolic above $\mathcal{P}$ of type i (i.e., the face corresponding to α_i and the vertex corresponding to $\mathcal{P}_i$ span the chamber for $\mathcal{P}$). Let $\chi_i : \mathcal{P} \rightarrow \mathfrak{Mult}$ be the restriction of the canonical character $\chi_{\mathcal{P}_i} : \mathcal{P}_i \rightarrow \mathfrak{Mult}$.

The set of roots $\{\alpha_1, \ldots, \alpha_r\}$ is a basis for $\mathrm{X}(\mathcal{P}) \otimes \mathbb{R}$ and so is the set $\{\chi_1, \ldots, \chi_r\}$. This determines real (in fact rational) numbers c_{ij} and n_{ji} such that:

$$\begin{aligned} \alpha_i &= \sum_j c_{ij}\chi_j \\ \chi_j &= \sum_i n_{ji}\alpha_i \end{aligned}$$

These two bases are almost dual. Let $\langle -, - \rangle$ be an inner product on $\mathrm{X}(\mathcal{P}) \otimes \mathbb{R}$ invariant under the action of the Weyl group. The α_i are simple roots and the χ_j point in the direction of the fundamental weights. Thus:

$$\begin{array}{rcll} 0 & \leq & n_{ji} & \\ 0 & < & n_{jj} & \\ 0 & < & c_{jj} & \\ \langle \chi_j, \alpha_i \rangle & = & 0 & \text{if } j \neq i \\ \langle \chi_j, \alpha_i \rangle & > & 0 & \text{if } j = i \\ \langle \alpha_i, \alpha_j \rangle & \leq & 0 & \text{for all } i, j \\ \langle \chi_j, \chi_k \rangle & \geq & 0 & \text{for all } j, k \end{array}$$

Observation 11.5. *Let A be a subset of*

$$\left\{ (\alpha_1, \ldots, \alpha_r, \chi_1, \ldots, \chi_r) \in \mathbb{R}^{2r} \;\middle|\; \chi_j = \sum_i n_{ji}\alpha_i \right\}$$

with the coefficients n_{ji} as above. The χ_j depend on the α_i. Hence, A is bounded if and only if its projection onto the first r coordinates is. Moreover, the coefficients n_{ji} are non-negative and strictly positive for $j = i$. Hence, χ_i tends to ∞ if α_i tends to ∞ while all other α_j stay bounded from below. Thus, the following are equivalent:

1. *A is bounded.*
2. *There exists constants* c_α^- *and* c_α^+ *with*

$$c_\alpha^- \le \alpha_i \le c_\alpha^+ \qquad \textit{for all } i$$

   for all $(\alpha_1, \dots, \alpha_r, \chi_1, \dots, \chi_r) \in A$.
3. *There exists constants* c_α^- *and* c_χ^+ *with*

$$c_\alpha^- \le \alpha_i, \;\; \chi_j \le c_\chi^+ \qquad \textit{for all } i, j$$

   for all $(\alpha_1, \dots, \alpha_r, \chi_1, \dots, \chi_r) \in A$. □

Recall, that $\mathcal{P}_j$ denotes the maximal parabolic of type j containing the minimal parabolic $\mathcal{P}$. Harder defines the invariants

$$\nu_i(\mathcal{P}, \mathcal{C}) := \prod_j \pi(\mathcal{P}_j, \mathcal{C})^{c_{ij}},$$

but we find it more convenient to express his results using logarithms:

$$\begin{aligned} \beta(\mathcal{P}_j, \mathcal{C}) &:= \log_{|k|}(\pi(\mathcal{P}_j, \mathcal{C})) \\ \mu_i(\mathcal{P}, \mathcal{C}) &:= \log_{|k|}(\nu_i(\mathcal{P}, \mathcal{C})) \end{aligned}$$

Note that $\pi(\mathcal{P}_j, \mathcal{C}) > 0$. Now, we have

$$\mu_i(\mathcal{P}, \mathcal{C}) = \sum_j c_{ij} \beta(\mathcal{P}_j, \mathcal{C});$$

and we say that a constant C_1 is a <u>lower reduction bound</u> if for any $\gamma \in \mathcal{G}(\mathbb{A})$, there exists a minimal K-parabolic subgroup $\mathcal{P}$ satisfying $\mu_i(\mathcal{P}, {}^\gamma\mathcal{G}(\mathcal{O})) \ge C_1$ for all i. In this language, the main theorems of reduction theory read as follows:

Theorem 11.6 ([Hard69, Satz 2.3.2]). *If* $\mathcal{G}$ *is* K*-isotropic it admits a lower reduction bound.* □

For a minimal parabolic $\mathcal{P}$ and an element $\gamma \in \mathcal{G}(\mathbb{A})$, we say that the parabolic <u>reduces</u> γ <u>with bound C_1</u>, if $\mu_i(\mathcal{P}, {}^\gamma\mathcal{G}(\mathcal{O})) \ge C_1$ for all i. We may not specify the bound if it is clear from the context.

Theorem 11.7 ([Hard69, Satz 2.3.3]). *Assume that* $\mathcal{G}$ *is* K*-isotropic. For any lower reduction bound* C_1 *there is another constant* C_2 *(which we call the* <u>*upper reduction bound*</u>*) such that: whenever* $\mathcal{P}$ *is a minimal* K*-parabolic reducing* $\gamma \in \mathcal{G}(\mathbb{A})$ *with bound* C_1 *and* $\mu_i(\mathcal{P}, {}^\gamma\mathcal{G}(\mathcal{O})) \ge C_2$ *then* any *minimal* K*-parabolic that reduces* γ *is contained in the maximal* K*-parabolic of type* i *above* $\mathcal{P}$. □

Theorem 11.8 (Mahler's Compactness Criterion). *Assume that $\mathcal{G}$ is K-isotropic. A subset $M \subseteq \mathcal{G}(\mathbb{A})^\circ$ is relatively compact modulo $\mathcal{G}(K)$ if any only if there are two constants c_- and c_+ such that for every $\gamma \in M$ there exists a minimal K-parabolic subgroup $\mathcal{P}$ with*

$$c_- \leq \mu_i(\mathcal{P}, {}^\gamma\mathcal{G}(\mathcal{O})) \leq c_+$$

for each i. Without loss of generality, the lower bound c_- can be taken to be any lower reduction bound.

It is a little unfortunate that Harder states Theorem 11.8 only in the case that $\mathcal{G}$ is K-split. Harder also provides the means of deducing the non-split case, but he does not carry out the argument. We provide an outline.

Proof of Theorem 11.8. For a fixed $\gamma \in \mathcal{G}(\mathbb{A})$, consider the set

$$\mathfrak{M} = \mathfrak{M}(\gamma) := \{\mu_i(\mathcal{P}, {}^\gamma\mathcal{G}(\mathcal{O})) \mid \mathcal{P} \text{ minimal } K\text{-parabolic reducing } \gamma,\ i \text{ arbitrary}\} \subseteq \mathbb{R}$$

We claim that this set is bounded. It is bounded from below since the parabolics $\mathcal{P}$ are assumed to reduce with respect to a fixed lower reduction bound C_1.

There are only finitely many types. So, assuming that $\mathfrak{M}$ is not bounded from above, there is a i such that

$$\mathfrak{M}_i := \{\mu_i(\mathcal{P}, {}^\gamma\mathcal{G}(\mathcal{O})) \mid \mathcal{P} \text{ minimal } K\text{-parabolic reducing } \gamma\}$$

is unbounded. Observation 11.5 then implies that

$$\mathfrak{B}_i := \{\beta(\mathcal{P}_i, {}^\gamma\mathcal{G}(\mathcal{O})) \mid \mathcal{P}_i \text{ maximal } K\text{-parabolic of type } i \text{ containing a } \mathcal{P} \text{ reducing } \gamma\}$$

is unbounded. This, however, contradicts Theorem 11.7. Hence, $\mathfrak{M}(\gamma)$ is bounded.

Now assume that $M \subseteq \mathcal{G}(\mathbb{A})^\circ$ is relatively compact modulo $\mathcal{G}(K)$. Then $\bigcup_{\gamma \in M} \mathfrak{M}(\gamma)$ is still bounded. The constant C_1 can be taken as c_- and the upper bound can be taken as c_+.

To argue the converse, we let K' be a finite Galois extension of K such that $\mathcal{G}$ is K'-split. Let M be a subset of $\mathcal{G}(\mathbb{A})^\circ$. By Lemma 11.1, M is relatively compact modulo $\mathcal{G}(K)$ in $\mathcal{G}(\mathbb{A})^\circ$ if it is relatively compact modulo $\mathcal{G}(K')$ in $\mathcal{G}(\mathbb{A}')^\circ$. As Harder argues in [Hard69, Lemma 2.2.2], this happens if there are two constants $c'_+ \geq c'_- > 0$ such that for each $\gamma \in M$ and each K'-Borel subgroup $\mathcal{B}$ the inequality

$$c'_- \leq \nu_{i,j}(\mathcal{B}, {}^\gamma\mathcal{K}') \leq c'_+$$

holds where $\mathcal{K}'$ is a suitable standard subgroup in $\mathcal{G}(\mathbb{A}')$.

Given bounds c_- and c_+ as in the statement of Theorem 11.8, one can find such c'_- and c'_+ using [Hard69, Lemma 2.3.5]. We remark that the exponent n in that statement is the degree of the extension K'/K, see [Hard69, Lemma 2.2.6]. □

The following alternate form of Mahler's compactness criterion is a consequence of Observation 11.5:

Corollary 11.9 (Mahler's Compactness Criterion, alternate form). *Let C_1 be a lower reduction bound. A subset $M \subseteq \mathcal{G}(\mathbb{A})^\circ$ is relatively compact modulo $\mathcal{G}(K)$ if any only if there is a constant c_+ such that for every $\gamma \in M$ there exists a minimal K-parabolic subgroup $\mathcal{P}$ that reduces γ with bound C_1 and satisfies $\beta(\mathcal{P}_j, {}^\gamma\mathcal{G}(\mathcal{O})) \leq c_+$ for each j.* □

12 Geometric reduction theory

In this section, $\mathcal{G}$ is assumed to be connected and reductive. The group $\mathcal{G}(K_p)$ acts on the associated euclidean Bruhat-Tits building X_p. The action is not necessarily type-preserving, but it is transitive on chambers; in particular, it has only finitely many orbits of vertices. The subgroup $\mathcal{G}(\mathcal{O}_p)$ is the stabilizer of some vertex. The group $\mathcal{G}(\mathbb{A}_S)$ acts componentwise on the product $X := \prod_{p \in S} X_p$ (components corresponding to places not in S act trivially). The subgroup $\mathcal{G}(\mathcal{O})$ is the stabilizer of some vertex $*$ in X. There are only finitely many $\mathcal{G}(\mathbb{A}_S)$-orbits of vertices in X; hence, there is a uniform upper bound for the distance of any point in X to the orbit $\mathcal{G}(\mathbb{A}_S) \cdot *$. Heuristically, the translation of reduction theory into the language of buildings proceeds via *pretending* that the euclidean building X can be identified with the orbit space $\mathcal{G}(\mathbb{A}_S) / \mathcal{G}(\mathcal{O})$.

To make this more precise, let x_p the vertex in X_p stabilized by the group $\mathcal{G}(\mathcal{O}_p)$, and let $\mathfrak{X}_p$ denote the $\mathcal{G}(K_p)$-orbit of x_p. Hence

$$\mathfrak{X}_p = \mathcal{G}(K_p) / \mathcal{G}(\mathcal{O}_p).$$

Putting $\mathfrak{X} := \prod_{p \in S} \mathfrak{X}_p$, we have:

$$\begin{aligned} \mathfrak{X} &= \prod_{p \in S} \mathcal{G}(K_p) / \mathcal{G}(\mathcal{O}_p) \\ &= \prod_{p \in S} \mathcal{G}(K_p) / \mathcal{G}(\mathcal{O}_p) \quad \times \quad \prod_{p \notin S} \mathcal{G}(\mathcal{O}_p) / \mathcal{G}(\mathcal{O}_p) \\ &= \mathcal{G}(\mathbb{A}_S) / \mathcal{G}(\mathcal{O}) \end{aligned}$$

Conversely, for any vertex $\mathfrak{x} \in \mathfrak{X}$, the stabilizer $\mathrm{Stab}(\mathfrak{x})$ in $\mathcal{G}(\mathbb{A}_S)$ is a standard subgroup of $\mathcal{G}(\mathbb{A})$.

We can now start to interpret reduction theory in terms of Busemann functions. Let Δ be the spherical building of $\mathcal{G}(K)$ over the global field, i.e., the simplicial complex that is the realization of the poset of proper K-parabolic subgroups of $\mathcal{G}$. Any vertex $v \in \mathcal{V}(\Delta)$ corresponds to a maximal K-parabolic $\mathcal{P}_v$ of $\mathcal{G}$. In particular, the building Δ is empty if and only if $\mathcal{G}$ is anisotropic over K. The anisotropic case is implicitly excluded in all considerations that require Δ to be non-empty. Note, however, that any statement of the form "for any vertex v in Δ, ..." is vacuously true.

For $v \in \mathcal{V}(\Delta)$, we define:

$$\begin{aligned} \tilde{\beta}_v : \mathfrak{X} &\longrightarrow \mathbb{R} \\ \mathfrak{x} &\mapsto \beta(\mathcal{P}_v, \mathrm{Stab}(\mathfrak{x})) \end{aligned}$$

We would like to show that $\tilde{\beta}$ can be extended to a Busemann function on X.

Let $\mathcal{P}$ be a minimal K-parabolic subgroup of $\mathcal{G}$ corresponding to a chamber c of Δ. By [Spri98, Theorem 13.3.6], the group $\mathcal{P}$ contains a maximal torus $\mathcal{T}$ that is defined over K. Of course, $\mathcal{T}$ is not necessarily split over K. Let $\mathcal{T}'$ be the maximal K-split subtorus of $\mathcal{T}$. For each place $p \in S$, let $\mathcal{T}'_p$ be the maximal K_p-subtorus of

$\mathcal{T}$. Note that $\mathcal{T}' \leq \mathcal{T}'_p$ for each $p \in S$. Let Σ_p be the apartment corresponding to $\mathcal{T}'_p$ in the euclidean building X_p. We put $\Sigma := \prod_{p\in S} \Sigma_p$ and $\mathfrak{S} := \mathfrak{X} \cap \Sigma$.

Lemma 12.1. *For any vertex $v \in c$, there exists an affine function on Σ that agrees with $\tilde{\beta}_v$ on the set $\mathfrak{S}$.*

Proof. From the Transformation Formula in Proposition 11.3 we obtain

$$\beta(\mathcal{P}_v, {}^{\gamma}\mathcal{C}) - \beta(\mathcal{P}_v, \mathcal{C}) = -\log_{|k|}(\|\chi_{\mathcal{P}_v}(\gamma)\|) = \sum_p d_p p(\chi_{\mathcal{P}_v}(\gamma)) \tag{11}$$

for each $\gamma \in \mathcal{T}(\mathbb{A})$. Considering this statement just for $\gamma \in \prod_{p\in S} \mathcal{T}'_p(K_p)$ with $\mathcal{C}$ taken to be the stabilizer of a vertex in $\mathfrak{S}$, the claim follows. □

So far, we cannot speak of Busemann functions on X as we did not yet fix a euclidean metric on X. There is some freedom in making this choice: on the one hand, we can freely rescale metrics on the factors X_p; on the other hand, the factors X_p need not be irreducible and if X_p decomposes as a product, the metrics on the irreducible factors can be independently scaled. However, that is the only source of non-uniqueness: up to scaling, there is a unique Weyl-group invariant metric on any irreducible euclidean building. In particular, we only have to choose the relative scales of the factors X_p if $\mathcal{G}$ is absolutely almost simple.

However, even in the case of an absolutely almost simple group and a single place $S = \{p\}$, we would have something to prove: we do not just want some metric on X. Rather, we would like a metric so that the geometry of the root system constructed in 11.4 is reflected in the angular metric at infinity induced from the euclidean metric on X. The reason for this restriction stems from the following: Harder's reduction theory is phrased in terms of the roots α_i and the dual characters χ_j. In our translation, we want to dispose of the roots α_i and only work with the characters (to those, our Busemann function will correspond). The euclidean metric is supposed to supply the necessary duality by means of its associated inner product. Thus, we need to demonstrate how this can be achieved.

As a first step, we compare the root system for $\mathcal{G}$ over the global field K to the root system over the local field K_p. If the field extension K_p/K was normal, we could directly quote [BoTi65, § 6]. Let K_p^{s} be the abstract separable closure of K_p, and let K^{s} be the separable closure of K inside K_p^{s}. Now K_p^{s}/K_p and K^{s}/K are both normal extensions. The group $\mathcal{G}$ splits over K^{s} and K_p^{s}. Moreover, the root systems for $\mathcal{G}$ over K^{s} and K_p^{s} are canonically isomorphic: a maximal K^{s}-split torus $\mathcal{T}^{\mathrm{s}}$ in $\mathcal{G}$ is also maximal K_p^{s}-split, and all its characters defined over K_p^{s} are already defined over K^{s}; hence

$$\mathrm{X}_{K^{\mathrm{s}}}(\mathcal{T}^{\mathrm{s}}) \otimes \mathbb{R} = \mathrm{X}_{K_p^{\mathrm{s}}}(\mathcal{T}^{\mathrm{s}}) \otimes \mathbb{R} =: V^{\mathrm{s}}. \tag{12}$$

Let $\mathcal{T}'$ be a maximal K-split torus in $\mathcal{T}^{\mathrm{s}}$ and put $V := \mathrm{X}_K(\mathcal{T}') \otimes \mathbb{R}$. Restriction of characters on $\mathcal{T}^{\mathrm{s}}$ to $\mathcal{T}'$ induces a projection $V^{\mathrm{s}} \to V$. We endow V^{s} with an inner product that is invariant under the full spherical Weyl group. By [BoTi65, § 6.10], there is a canonical way of realizing the abstract vector space V as a subspace of V^{s}

such that the orthogonal projection is the restriction homomorphism. The induced inner product on V is invariant under the Weyl group of $\mathcal{G}$ over K.

The same construction can be carried out for K_p, yielding a subspace V_p of V^{s}. We want to argue the inclusion $V \leq V_p$. To do so, we impose the assumption that the chosen K^{s}-split torus $\mathcal{T}^{\mathrm{s}}$ is defined over K. It is then also defined over K_p. Moreover, [BoTi65, § 6.11] applies: the vector space V is the fixed point set of the Galois action of $\mathrm{Gal}(K^{\mathrm{s}}/K)$ on V^{s}. Similarly, V_p is the fixed point set of $\mathrm{Gal}\big(K_p^{\mathrm{s}}/K_p\big)$ on V^{s}. Since any K-automorphism of K_p^{s} leaves the separable closure $K^{\mathrm{s}} \subseteq K_p^{\mathrm{s}}$ invariant, we have a homomorphism

$$\mathrm{Aut}\big(K_p^{\mathrm{s}}/K\big) \to \mathrm{Gal}(K^{\mathrm{s}}/K)$$

and the group $\mathrm{Gal}\big(K_p^{\mathrm{s}}/K_p\big)$ acts on V^{s} via this projection (here, we consider the identification made in 12). Hence $V \leq V_p$.

In particular, these considerations apply to the situation discussed in Lemma 12.1. In that case, $\mathcal{T}^{\mathrm{s}}$ is just the maximal K-torus $\mathcal{T}$ within the minimal parabolic $\mathcal{P}$. $\mathcal{T}$ automatically splits over the separable closures K^{s} and K_p^{s}.

Proposition 12.2. *Assume that $\mathcal{G}$ is a connected reductive group. There exists a euclidean metric on $X = \prod_{p \in S} X_p$ and for each vertex $v \in \mathcal{V}(\Delta)$, there is a positive coefficient $s_j \in \mathbb{R}$, depending only on the type $j = \mathrm{type}(v)$, such that the following hold:*

1. *For each vertex $v \in \mathcal{V}(\Delta)$, the rescaled function $s_j \tilde{\beta}_v : \mathfrak{X} \to \mathbb{R}$ is the restriction of a Busemann function $\beta_v : X \to \mathbb{R}$ to $\mathfrak{X}$. Let $e_v \in \partial(X)$ be the center of the Busemann function, i.e., the visual end point of the gradient of β_v.*

2. *For each vertex $v \in \mathcal{V}(\Delta)$, the Busemann function β_v is non-constant on each factor X_p of X. In particular if all factors X_p are irreducible (e.g., if $\mathcal{G}$ is absolutely almost simple), the Busemann functions β_v are in general position.*

3. *The map $v \mapsto e_v$ induces an isometric embedding of Δ into $\partial(X)$.*

4. *For each $\gamma \in \Gamma$, each vertex $v \in \mathcal{V}(\Delta)$, and each point $x \in X$, we have $\beta_{\gamma v}(\gamma x) = \beta_v(x)$. In particular, the map $v \mapsto e_v$ and the induced embedding $\Delta \hookrightarrow \partial(X)$ are Γ-invariant.*

Proof. If $\mathcal{G}$ is anisotropic over K, the building Δ is empty, and there is nothing to prove: the proposition is vacuously true. So, we assume that $\mathcal{G}$ is K-isotropic.

We choose $\mathcal{T}$ as our standard apartment Σ for X, i.e., within each factor X_p the standard apartment Σ_p corresponds to the maximal K_p-split torus $\mathcal{T}'_p$ within $\mathcal{T}$. We fixed an inner product on V^{s} and this inner product induces inner products on each V_p, which is the metric model for the standard apartment Σ_p. This way, we define a metric on Σ and thus on the euclidean building X.

By Lemma 12.1, the function $\tilde{\beta}_v$ agrees with an affine function on Σ. Affine functions on euclidean spaces are Busemann functions up to rescaling. Hence, we can choose a factor s_j so that the rescaled function $s_j \tilde{\beta}_v$ agrees with a Busemann function $\beta_v : X \to \mathbb{R}$ on Σ. By Equation 11 from the proof of Lemma 12.1, the rescale factor

s_j depends only on the length of the canonical character $\chi_{\mathcal{P}_v}$ in the product $\prod_{p\in S} V_p$. Hence, s_j depends only on the type of v.

Again by Equation 11, the center of β_v is given be the visual end of the fundamental weight. In particular, it is stabilized by $\mathcal{P}(\mathbb{A}_S)$. As $\mathcal{G}(K_p)$ acts strongly transitively on the factor X_p, translates of Σ_p under the action of the parabolic $\mathcal{P}(K_p)$ cover X_p. Hence, the $\mathcal{P}(\mathbb{A}_S)$-translates of Σ cover X. The Transformation Formula 10 implies that $s_j\tilde{\beta}_v$ agrees with β_v on each of these translates. This proves the first claim.

The second claim follows directly from Equation 11 since $d_p \neq 0$ for each $p \in S$.

Claim 3 follows from the discussion preceding this proposition. The geometry of the chamber c of Δ corresponding to the minimal parabolic $\mathcal{P}$ (i.e., the angular distances between its vertices) is given by the angles between the fundamental weights in V. As $V \leq V_p$, we have an induced diagonal embedding of V into the orthogonal product $\prod_{p\in S} V_p$. This is an isometric embedding modeling the map $v \mapsto e_v$ on the standard apartment. Considering other minimal K-parabolics in the same apartment, we see that $\Delta \hookrightarrow \partial(X)$ is an isometric embedding on the standard apartment. The choice of the standard apartment was arbitrary and does not influence the embedding. Hence, $\Delta \hookrightarrow \partial(X)$ is an isometric embedding on each apartment. Since any two points of Δ are contained in a common apartment, $\Delta \hookrightarrow \partial(X)$ preserves distances.

Finally, claim 4 follows from Equation 9. As Γ might not act type-preserving, we have to consider the rescaling factors s_j and $s_{\gamma v}$. These factors depend only on the length of the associated canonical characters. As Γ acts by isometries, $s_j = s_{\gamma v}$. □

Let c be a chamber of Δ and let $\mathcal{P}$ denote the corresponding minimal K-parabolic. Let Σ be an apartment of X with $c \subseteq \partial(\Sigma)$. The Busemann functions β_v associated with vertices $v \in c$ restrict to affine functions on Σ. Put:

$$\Sigma_0 := \{x \in \Sigma \mid \beta_v(x) = 0 \text{ for each } v \in c\}$$

The metric on Σ is constructed from a metric on $\mathrm{X}_K(\mathcal{P})\otimes\mathbb{R}$ invariant under the action of the Weyl group; and the quotient Σ/Σ_0 is isometric to $\mathrm{X}_K(\mathcal{P})\otimes\mathbb{R}$. The Busemann functions β_v descend to the quotient Σ/Σ_0. They form a system of coordinates, which under the isometry $\Sigma/\Sigma_0 \cong \mathrm{X}_K(\mathcal{P})\otimes\mathbb{R}$ corresponds to the set of fundamental weights up to rescaling. Recall that the simple roots are related to the rescaled fundamental weights by the matrix (c_{ij}). Thus, we define:

$$\mu_i^c := \sum_{v\in c} \frac{c_{i\,\mathrm{type}(v)}}{s_{\mathrm{type}(v)}}\beta_v : X \longrightarrow \mathbb{R}$$

Observation 12.3. *For any $\mathfrak{x} \in \mathfrak{X} \subseteq X$, we have $\mu_i^c(\mathfrak{x}) = \mu_i(\mathcal{P}, \mathrm{Stab}(\mathfrak{x}))$.* □

Hence Theorem 11.6, the first main theorem of Harder's reduction theory, implies:

Corollary 12.4. *If $\mathcal{G}$ is K-isotropic, there is a constant $C_1 \in \mathbb{R}$ such that for any point $x \in X$, there exists a chamber c in Δ with $\mu_i^c(x) \geq C_1$.* □

Restricted to Σ, the functions μ_i^c are affine and the duality between fundamental weights and simple roots translates into the following relationship:

Observation 12.5. *For any real number $t \in \mathbb{R}$ and any face $\tau \subseteq c$, we consider the convex cone (with tip parallel to Σ_0):*

$$Y_{\Sigma,\tau}(t) := \{x \in \Sigma \,|\, \beta_v(x) \le t \text{ for each } v \in \tau\}$$

We also define:

$$Z_{\Sigma,\tau}(t) := \left\{x \in \Sigma \,\middle|\, \mu_{\mathrm{type}(v)}(x) \ge t \text{ for each } v \in \tau\right\}$$

Then, $Z_{\Sigma,\tau}(0)$ is the normal cone for $Y_{\Sigma,\tau}(0)$; i.e., $Z_{\Sigma,\tau}(0)$ consists precisely of those points in Σ whose closest point projection onto $Y_{\Sigma,\tau}(0)$ lies in the tip Σ_0. □

Let

$$\mathrm{pr}^t_{\Sigma,\tau} : \Sigma \longrightarrow Y_{\Sigma,\tau}(t)$$

denote the closest point projection. As seen in Observation 1.1, for $x \in \Sigma$, the value $b^t_{\tau,v}(x) := \beta_v\big(\mathrm{pr}^t_{\Sigma,\tau}(x)\big)$ is independent of the apartment Σ. Recall the definition

$$\sigma_t(x,\tau) := \left\{v \in \tau \,\middle|\, b^t_{\tau,v}(x) = t\right\}.$$

Also recall that x is t-reduced by c if $\sigma_t(x,c) = c$. The closest point projection is embedded into this terminology so that it allows us to characterize normal cones to $Y_{\Sigma,c}(t)$.

Observation 12.6. *The set*

$$N_{\Sigma,c}(t) := \{x \in \Sigma \,|\, x \text{ is } t\text{-reduced by } c\}$$

is the normal cone to $Y_{\Sigma,c}(t)$. In particular, it is a translate of $Z_{\Sigma,c}(0)$. Thus, there exist real constants $t'_1, \ldots, t'_r$ only depending on t such that

$$\{x \in \Sigma \,|\, x \text{ is } t\text{-reduced by } c\} = \{x \in \Sigma \,|\, \mu^c_i(x) \ge t'_i \text{ for each } i = 1, \ldots, r\}.$$ □

If Σ' is another apartment of X whose visual boundary contains the chamber c, then the isomorphism of Coxeter complexes $\iota : \Sigma \to \Sigma'$ from Observation 1.1 does not only commute with the Busemann functions β_v but also with the functions μ^c_i. In particular, it identifies $Z_{\Sigma,c}(t)$ with $Z_{\Sigma',c}(t)$ and $N_{\Sigma,c}(t)$ with $N_{\Sigma',c}(t)$. We define:

$$\begin{aligned}
N_c(t) &:= \bigcup_{\Sigma\,:\,c\subseteq\partial(\Sigma)} N_{\Sigma,c}(t) \\
&= \{x \in X \,|\, x \text{ is } t\text{-reduced by } c\} \\
Z_c(t') &:= \bigcup_{\Sigma\,:\,c\subseteq\partial(\Sigma)} Z_{\Sigma,c}(t')
\end{aligned}$$

The systems $N_{\Sigma,c}(t)$ and $Z_{\Sigma,c}(t')$ are strongly related:

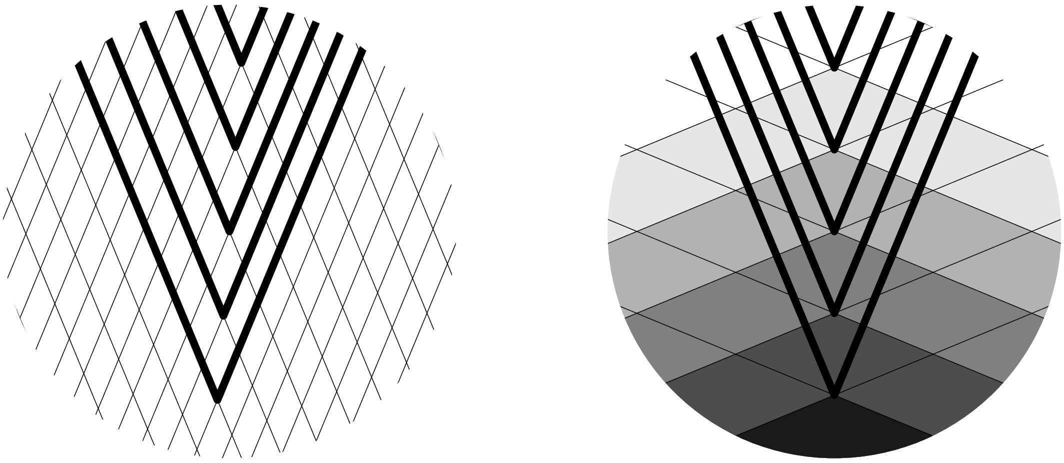

Figure 14: normal cones

The family $Z_{\Sigma,\tau}(-)$, shown on the left, is defined via the functions μ_j. Some level sets of the μ_j are drawn. In contrast, the family $N_{\Sigma,c}(-)$, shown on the right, is defined in terms of the Busemann functions β_v. The latter are normalized to have unit length gradient with respect to the metric. Their level sets are shown. The shaded areas indicate the family $Y_{\Sigma,\tau}(-)$.

Observation 12.7. *For any t there exist t'_+ and t'_- such that*

$$Z_{\Sigma,c}\big(t'_-\big) \subseteq N_{\Sigma,c}(t) \subseteq Z_{\Sigma,c}\big(t'_+\big)$$

for any apartment Σ and any K-rational chamber c in the visual boundary of Σ.

Analogously, for any t' there exist t_+ and t_- such that

$$N_{\Sigma,c}(t_-) \subseteq Z_{\Sigma,c}(t') \subseteq N_{\Sigma,c}(t_+)$$

for any $c \in \Delta$ and any euclidean apartment Σ whose visual boundary contains c. □

We can also relate the subsets $N_c(t)$ to their Hausdorff neighborhoods. For any subset $V \subseteq X$, let $\mathrm{Nbhd}_L(V)$ denote the Hausdorff neighborhood of radius L around V, i.e., the set of points in X of distance at most L to V.

Observation 12.8. *For any fixed distance $L \geq 0$, there is a constant C such that*

$$\mathrm{Nbhd}_L(N_c(t)) \subseteq N_c(t - C)$$

for any $t \in \mathbb{R}$ and any chamber c in Δ.

Conversely, for any given C there exists a constant L such that

$$N_c(t - C) \subseteq \mathrm{Nbhd}_L(N_c(t))$$

for any $t \in \mathbb{R}$ and any chamber c in Δ. □

We are ready for the geometric version of Theorem 11.6.

Proposition 12.9. *Assume that $\mathcal{G}$ is K-isotropic. For any fixed diameter $d \in \mathbb{R}$ there exists a constant $r \in \mathbb{R}$ such that for any $x \in X$ there is a chamber c that r-reduces each point y of distance at most d to x.*

Proof. By Corollary 12.4, there is a constant C_1 such that

$$X = \bigcup_{c \in \mathcal{C}(\Delta)} Z_c(C_1)\,.$$

By Observation 12.7, there is a bound r' such that

$$X = \bigcup_{c \in \mathcal{C}(\Delta)} N_c(r')\,.$$

Now, one chooses r so that $N_c(r)$ contains the d-Hausdorff neighborhood of $N_c(r')$ for any chamber c in Δ. □

The main theorem of geometric reduction theory reads as follows:

Theorem 12.10. *Assume that $\mathcal{G}$ is connected, reductive, and defined and isotropic over the global function field K. For any diameter d there exist constants r and R such that $(\beta_v : X \to \mathbb{R})_{v \in \mathcal{V}(\Delta)}$ together with the constants r and R is a d-uniform and Γ-invariant reduction datum. Moreover, for any $x \in X$ and any chamber c in Δ that r-reduces x, the simplex $\sigma_R(x, c)$ is contained in any chamber c' that r-reduces x.*

Proof. Using Observation 12.7, choose C_1 so that $N_c(r) \subseteq Z_c(C_1)$ for any c in Δ. Then, C_1 is a lower reduction bound.

We can be a little more specific: Let $\mathcal{P}$ be the minimal K-parabolic corresponding to a chamber c that r-reduces the point $\gamma *$ for some $\gamma \in \mathcal{G}(\mathbb{A}_S)$. Then $\mu_i(\mathcal{P}, {}^{\gamma}\mathcal{G}(\mathcal{O})) \geq C_1$ for all i.

By Theorem 11.7, there is a corresponding C_2. Using again Observation 12.7, we find a constant R such that $N_c(R) \subseteq Z_c(C_2)$ for all chamber c in Δ. Then, for any vertex $v \in c$ and any $x \in X$ that is r-reduced by c, we have:

$$v \in \sigma_R(x, c) \quad \Longrightarrow \quad \mu^c_{\mathrm{type}(v)}(x) \geq C_2$$

For $x = \gamma *$, it follows that the maximal K-parabolic $\mathcal{P}_v$ corresponding to v contains any minimal K-parabolic $\mathcal{P}'$ whose chamber c' r-reduces $\gamma *$. Hence, $\sigma_R(\gamma *, c) \subseteq c'$.

Extending coverage from $\mathfrak{X}$ to all of X requires changing the constants r and R only by a little. Hence, we have established a reduction datum.

We have already argued in Proposition 12.9 that this reduction datum is d-uniform. That it is Γ-invariant follows from Proposition 12.2, part 4. □

It remains to discuss Γ-cocompactness of the reduction datum. We consider the filtration of X by subspaces

$$Y_t := \{x \in X \mid \beta_v(x) \leq t \text{ for all } c \text{ reducing } x \text{ and all } v \in c\}\,.$$

Theorem 12.11. *If $\mathcal{G}$ does not admit any non-trivial K-characters, then Y_t has compact quotient modulo Γ. If there is a non-trivial K-character $\mathcal{G} \to \mathfrak{Mult}$, then Y_t does not have a compact quotient modulo Γ unless Y_t is empty.*

Proof. If there are no non-trivial characters, we have $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{A})^{\circ}$. Hence, cocompactness of the Y_t is immediate from Mahler's Compactness Criterion in its alternate form 11.9.

If there is a non-trivial character on $\mathcal{G}$, then $\mathcal{G}$ has a central K-torus. Corresponding to this torus, the euclidean building X has a euclidean space as a factor. Dirichlet's unit theorem implies that the S-arithmetic subgroup does not act cocompactly in the direction of this factor. □

In particular, an absolutely almost simple non-commutative group $\mathcal{G}$ does not admit non-trivial K-characters. Hence, Theorem 1.9 follows from Theorem 12.10 and Theorem 12.11.

Remark 12.12. Formally, the filtration is meaningful even for anisotropic $\mathcal{G}$. In that case, $Y_t = X$ independent of t. Theorem 11.2 implies that X/Γ is cocompact in this case. This way, one recovers Serre's proof that $\mathcal{G}(\mathcal{O}_S)$ is of type F_∞ for K-anisotropic $\mathcal{G}$ [Serr71, Cas (b), p. 126–127].

Remark 12.13. If $\mathcal{G}$ is K-isotropic and non-commutative, then Δ is non-empty and Corollary 1.8 yields an alternative description of the filtration:

$$Y_t = \{x \in X \mid \beta_v(x) \le t \text{ for some } c \text{ reducing } x \text{ and all } v \in c\} \qquad \text{for } t \ge R \qquad \square$$

13 The structure of the quotient X/Γ

Let $\mathcal{G}$ be a connected reductive K-group. To simplify the exposition, we assume in this section that $\mathcal{G}$ does not admit a non-trivial K-character so that the filtration Y_t is Γ-cocompact by Theorem 12.11. Our goal is to show that in this case the image Γ_0 of Γ in the automorphism group $\mathrm{Aut}(X)$ of the euclidean building is a lattice in $\mathrm{Aut}(X)$.

It is well-known that Γ_0 is a lattice. For a Chevalley group scheme $\mathcal{G}$, Harder [Hard69, page 41] constructs a fundamental set for $\mathcal{G}(K)$ in $\mathcal{G}(\mathbb{A})$. The restriction to Chevalley group schemes is unnecessary. In the same paper, Harder generalizes the main statements of reduction theory (also first proved for Chevalley groups) to arbitrary reductive groups. His construction of a fundamental domain can then be carried out in the same vein – in fact, Harder points out this possibility [Hard69, page 51]. He also remarks that the same argument as in [Gode63] following Théorèm 7 then shows that $\mathcal{G}(K)$ is a lattice in $\mathcal{G}(\mathbb{A})$. Harder also indicates [Hard69, pages 51ff] how this argument can be adapted to deal with the lattice $\mathcal{G}(\mathcal{O}_S)$ in $\mathcal{G}(\mathbb{A}_S)$. Since $\mathcal{G}(\mathbb{A}_S)$ acts cocompactly on X, the result shows that Γ_0 is a lattice in $\mathrm{Aut}(X)$.

Hence, the point of this section is to demonstrate that Theorem 12.10 preserves the necessary information. In fact, even the rough strategy of the proof is the same:

in Proposition 13.6 a fundamental set for Γ_0 in X is constructed; an application of Serre's criterion in the proof of Proposition 13.8 replaces the covolume estimate following [Gode63, Théorèm 7].

Remark 13.1. The exact covolume of $\mathcal{G}(K)$ in $\mathcal{G}(\mathbb{A})$ is not known in general. See [BeDh09] for conjectural values and partial results.

Observation 13.2. *The kernel of the projection $\Gamma \to \Gamma_0$ is the kernel of the action of Γ on X. Hence, it is finite as it is clearly contained in any vertex stabilizer of Γ, which is finite.* □

Observation 13.3. *If $\mathcal{G}$ is K-anisotropic, the group Γ_0 is a lattice. In fact, the quotient X/Γ_0 is compact by Theorem 12.11 or by Theorem 11.2.* □

Let r and R be constants satisfying Theorem 12.10.

Lemma 13.4. *For any point $x \in X$ there is at most one chamber c in Δ with $x \in N_c(R)$.*

Proof. Assume $x \in N_c(R)$. Then $\sigma_R(x, c) = c$. Hence, c is contained in any chamber c' that r-reduces x. In particular, c is the only chamber that r-reduces x, whence it is the only chamber that R-reduces x. □

Corollary 13.5. *For any point $x \in N_c(R)$, the set $N_c(R)$ is invariant under the induced action of the stabilizer $\mathrm{Stab}_{\Gamma_0}(x)$.* □

The following theorem provides a fundamental set for the action of Γ on X. One should compare it to [Serr80, Theorem 9, page 106].

Proposition 13.6. *Assume that $\mathcal{G}$ is K-isotropic and non-commutative. There exists a constant L, finitely many points $x_1, \ldots, x_s$, and as many chambers $c_1, \ldots, c_s$ in Δ such that the following hold:*

1. *The point x_i is R-reduced by c_i for each $i \in \{1, \ldots, s\}$. In particular, the union S_i of rays from x_i with visual endpoint in c_i is isometric to a flat sector.*
2. *Every point in X is within distance L to the orbit of some sector S_i. Equivalently, the Γ_0-translates of*
$$D := \mathrm{Nbhd}_L\left(\bigcup_{i=1}^{s} S_i\right)$$
cover X.
3. *For $i \neq j$, the Γ_0-orbits of S_i and S_j are disjoint.*

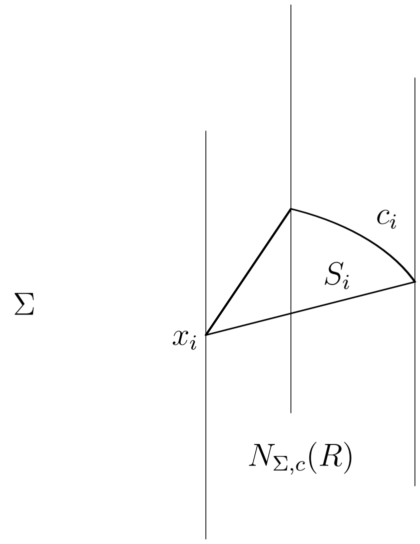

Figure 15: a wedge $N_{\Sigma,c}(R)$ and a sector S_i
Here, the rational building Δ has dimension 1 and the euclidean building X has dimension 3.

Proof. The set $\bigcup_{c\in\mathcal{C}(\Delta)} N_c(R)$ is Γ_0-invariant. Hence $Q := Y_R \cap \bigcup_{c\in\mathcal{C}(\Delta)} N_c(R)$ is Γ_0-invariant with compact quotient. By Lemma 13.4, each point $x \in Q$ has a unique R-reducing chamber in Δ. Moreover, this chamber varies Γ_0-equivariantly with the point. Since Q/Γ_0 is compact, there exist a constant L_0, finitely many pairwise Γ_0-inequivalent chambers $c_1, \ldots, c_s$ in Δ, and points $x_1, \ldots, x_s$ such that the following holds:

> For each point $x \in Q$ there exists a $\gamma \in \Gamma_0$ and a unique index $i \in \{1, \ldots, s\}$ such that x is within distance at most L_0 of γx_i and so that x is R-reduced by the chamber γc_i.

Let S_i be the union of geodesic rays from x_i with visual endpoint in c_i. We put

$$D_0 := \mathrm{Nbhd}_{L_0}\left(\bigcup_{i=1}^{s} S_i\right)$$

and claim that the Γ_0-translates of D_0 cover the union $\bigcup_{c\in\mathcal{C}(\Delta)} N_c(R)$. Recall that $N_c(R)$ is the union of *wedges* $N_{\Sigma,c}(R)$. The tip of such a wedge consists precisely of the subspace $\{x \in \Sigma \mid \beta_v(x) = R \text{ for all } v \in c\}$, which is a subset of Q. Now, let $y \in N_{\Sigma,c}(R)$. Then, there is a unique point $x \in Q \cap N_{\Sigma,c}(R)$ such that y lies on a geodesic ray from x with visual endpoint in c. There is $\gamma \in \Gamma_0$ and a subscript $i \in \{1, \ldots, s\}$ such that $c = \gamma c_i$ and such that x is within distance at most L of γx_i. It follows that y is within distance L of γS_i.

By Theorem 12.9, $X = \bigcup_c N_c(r)$, and by Observation 12.8 there is a constant L_1 such that $N_c(r) \subseteq \mathrm{Nbhd}_{L_1} N_c(R)$ for all chambers c of Δ. It follows that $L := L_0 + L_1$ yields a domain

$$D := \mathrm{Nbhd}_{L}\left(\bigcup_{i=1}^{s} S_i\right)$$

whose Γ_0-translates cover X. □

Lemma 13.7. *For every distance L there is a constant K such that for any two points $x, y \in X$ of distance at most L the inequality*

$$\frac{1}{K} \left|\mathrm{Stab}_{\Gamma_0}(x)\right| \leq \left|\mathrm{Stab}_{\Gamma_0}(y)\right| \leq K \left|\mathrm{Stab}_{\Gamma_0}(x)\right|$$

holds.

Proof. The ball $B := B_L(y)$ of radius L with center y is invariant under the induced action of $\mathrm{Stab}_{\Gamma_0}(y)$. The group Γ_0 acts by cell-permuting homeomorphisms on X, hence the induced action on the barycentric subdivison is rigid: a cell that is stabilized is fixed pointwise. By restriction, the action of $\mathrm{Stab}_{\Gamma_0}(y)$ on the induced cell decomposition of B inherits this property. In particular, the size of any orbit is bounded by the number K_y of cells in B.

The index of the group $\mathrm{Stab}_{\Gamma_0}(x) \cap \mathrm{Stab}_{\Gamma_0}(y)$ in $\mathrm{Stab}_{\Gamma_0}(y)$ is given by the size of the orbit $\mathrm{Stab}_{\Gamma_0}(y) \cdot x$. Hence, it is bounded by K_y.

The claim follows since there is a maximum number of cells that a ball of radius L centered anywhere in X can meet. □

Proposition 13.8. *The group Γ_0 is a lattice in* $\mathrm{Aut}(X)$.

Proof. We consider the action of $\mathrm{Aut}(X)$ on the set $\mathcal{C}(X)$ of chambers of X. By [BaLu01, 1.6 Corollary], we have to argue that the infinite sum

$$\sum_{C \in \mathcal{X}} \frac{1}{\left|\mathrm{Stab}_{\Gamma_0}(C)\right|} \tag{13}$$

converges, where $\mathcal{X}$ is a set of representatives of $\mathcal{C}(X)/\Gamma_0$.

Let L, $x_1, \ldots, x_s$, $c_1, \ldots, c_s$, $S_1, \ldots, S_n$, and D be as in Proposition 13.6. We now choose $\mathcal{X}$ to be the collection of all chambers in X that intersect D.

The set $Q_i := Y_R \cap N_{c_i}(R)$ consists of the tips of those wedges that form $N_{c_i}(R)$. Let ρ be some geodesic ray from x_i in S_i, i.e., the visual endpoint e of ρ lies in c_i. Let T_ρ be the union of all geodesic rays in X that share an infinite segment with ρ. Then T_ρ is a locally finite tree and intersects Q_i in a discrete set. Let $\mathcal{R}_\rho$ be the set of points in $Q_i \cap T_\rho$ that lie within distance L of x_i.

Consider a point y on the ray ρ. The union of all geodesic rays in X with endpoint e that pass through y is a subtree T_y of T_ρ. The intersection $T_y \cap Q_i$ is a finite set, on which the finite group $\mathrm{Stab}_{\Gamma_0}(y)$ acts. By Proposition 13.6, each $\mathrm{Stab}_{\Gamma_0}(y)$-orbit has a representative in $\mathcal{R}_\rho$. As the cardinality $|Q_i \cap T_y|$ grows exponentially with the distance $\mathrm{dist}(y, x_i)$, so does the size $|\mathrm{Stab}_{\Gamma_0}(y)|$. As c_i is compact, the growth rate is uniformly bounded away from 1 for all rays ρ from the tip x_i into S_i.

On the other hand, the number of chambers in X intersecting S_i at a point of distance d to x_i grows only polynomially with the distance d.

Finally, Lemma 13.7 shows that stabilizers of chambers of bounded distance have comparable sizes.

It follows that the sum (13) converges. □

The case that X is a tree is treated in [Serr80, Exercise 2 a, page 110].

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